DOCUMENT RESUME

ED 342 648 SE 052 502

AUTHOR	Resnick, Lauren B.
TITLE	From Protoquantities to Operators: Building Mathematical Competence on a Foundation of Everyday Knowledge.
INSTITUTION	Pittsburgh Univ., Pa. Learning Research and Development Center.
PUB DATE	Mar 91
NOTE	96p.
PUB TYPE	Information Analyses (070) -- Guides - Classroom Use - Teaching Guides (For Teacher) (052)
EDRS PRICE	MF01/PC04 Plus Postage.
DESCRIPTORS	Addition; Early Experience; Elementary Education; Elementary School Mathematics; *Intuition; *Learning Processes; Learning Theories; *Mathematical Applications; Mathematics Education; *Mathematics Instruction; *Models; Multiplication; *Problem Solving; Subtraction
IDENTIFIERS	*Informal Knowledge; Learning Research and Development Center

ABSTRACT

This paper proposes a theory that can account for differences between everyday and formal mathematics knowledge and a set of processes by which informal knowledge is transformed into formal mathematics. After an introduction, the paper is developed in five sections. The first section lays out the nature of informal, everyday mathematics knowledge. Examples of children's thinking for several mathematical principles are presented. The second section discusses evidence of systematic difficulty in learning school mathematics, considering two hypotheses to explain why strong and reliable intuitions documented for children and unschooled people are not sustained in school mathematics. The first, called the "syntax-semantics" hypothesis, is that the focus in school on formal symbol manipulation discourages children from bringing their intuitions to bear on school mathematics learning. The second, called the "abstract entities" hypothesis, is that there is an epistemological discontinuity between informal mathematics rooted in everyday behavior and the kind of mathematical reasoning that is sought in school. The third section discusses a theory of layers of mathematical knowledge. The layers are characterized as the mathematics of protoquantities, quantities, numbers, and operators. The fourth section discusses teaching mathematics on the basis of intuitive knowledge, arranging situations of practice and discussion that will help children elaborate their schemas at successively higher levels of mathematical knowledge. The final section presents six principles that guided the development of a progam for elementary classroom instruction which not only produced substantial gains in children's mathematical performance but challenged some long-held assumptions in psychology and education as well. (MDH)

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In press

In G. Leinhardt, R. Putnum, & R. Hattrup (Eds.), Analyses of arithmetic for mathematics teachers.
Hillsdale, NJ: Erlbaum.

From Protoquantities to Operators: Building Mathematical Competence on a Foundation of Everyday Knowledge[1]

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Over the past decade two lines of research on mathematics learning have produced apparently contradictory results. One has documented substantial amounts of mathematical knowledge on the part of young children and minimally schooled adults. The other has documented persistent and systematic difficulties that many children have in learning school mathematics. Both lines of work are independently convincing. Taken together, however, they constitute a paradox of major proportions. How can it be that mathematics is simultaneously so ubiquitous and fundamental that everyone seems to learn it, and so difficult that many seem never to master it? What kinds of discontinuities could be producing this massive misfit?

It has become common, even fashionable, to blame the difficulties that students have in formal mathematics learning on the schools' active suppression of the "mathematics of the streets." The proposal seems to be that, if the schools would make the classroom more like the streets (with "real" problems to solve, social supports, calculation tools, and the like) everyone would learn math easily and well. Some seem to make this claim on the assumption that formal mathematics is in all essentials continuous with street math, so that there is no reason not to continue indefinitely with an applied problem solving program and expect substantial formal mathematical sophistication to eventually develop. Others make an explicit distinction between formal mathematics and everyday "applied math," but are fundamentally disinterested in the formal side.

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[1] This paper constitutes a five-year report on the Formal and Intuitive Knowledge in School Mathematics project of the Center for the Study of Learning, supported by the Office of Educational Research and Improvement at the Learning Research and Development Center, University of Pittsburgh.

March 11, 1991

The proposal that, because street math is easy and universal, the route to higher and more widely distributed mathematics achievement is to make the classroom more like the street, warrants serious consideration. Surely if learning proceeds effectively in non-school environments it is a worthwhile venture to try to figure out what makes those environments so effective, and perhaps mimic aspects of them in school. But the proposal as stated seems too simple. It fails to ask what is *not* learned well "on the streets" and what particular forms of learning might proceed best in environments specifically designed for the purpose. In the case of mathematics, in particular, the standard formulation overlooks the complexities of the many different kinds of knowledge that constitute mathematical competence and the relations between these kinds of knowledge.

In this paper I first lay out the nature of informal, everyday mathematics knowledge and consider two hypotheses that might underlie persistent difficulty in learning school mathematics. I then develop a theory that can account for differences between everyday and formal mathematics knowledge and describe a set of processes by which informal knowledge is transformed into formal mathematics. Finally, I consider what this empirically grounded epistemological analysis suggests for how elementary mathematics education might proceed. The arguments I develop are based on a multi-year program of research on the nature of informal mathematics knowledge and on a more recent project, developed in collaboration with a primary school teacher, that uses these research findings to design a radically changed classroom instructional program. This instructional program has been heavily influenced, as well, by recent research and theory that is challenging traditional views of knowledge, learning and teaching and calling for a reconceptualization of learning as apprenticeship in a particular environment of practice.

Additive Composition: Early and Universal Mathematics Knowledge

Much of elementary arithmetic has as its conceptual base the fact that all numbers are compositions of other numbers. This compositional character of numbers provides an intuitive basis for understanding fundamental properties of the number system. These properties include commutativity and associativity of addition, distribution of multiplication over addition, complementarity of addition and subtraction (additive inverse), and equivalence classes of addition pairs. Children appreciate these principles at a surprisingly young age, as I shall document in this chapter. My evidence is drawn primarily from a series of studies of invented arithmetic performances by children. Challenged to solve problems for which they have no ready algorithms, children invent procedures that can be shown to implicitly apply principles based on additive composition. Others have shown that similar reasoning takes place among minimally schooled adults carrying out arithmetic tasks as part of their daily work. Together this body of research points to a body of mathematical knowledge that appears to be easily, and probably universally, acquired.

Permissions and constraints in arithmetic procedures

The cases presented here will be analyzed in terms of the *permissions* and *constraints* on number operations that the additive composition principles embody. I can best illustrate the ways in which permissions and constraints interact to define a rule system, and the ways in which they derive from elementary principles of additive composition, by developing a justification for the standard subtraction-with-borrowing algorithm taught in American schools.

In multidigit subtraction the goal of the entire process is to find a difference between two quantities, each of which is symbolized by a string of digits that conform to the conventions of place value notation. Place value notation uses the additive composition principle to *permit* us to write an infinite set of natural numbers without needing an infinite number of distinguishable symbols. It does this by assigning a value to each position in an ordered string, so that an individual digit's values are determined by its position. This means that 324, for example, must be interpreted as a composition of 300 (itself a composition of 100 plus 100 plus 100) plus 20 (10 plus 10) plus 4 (1 plus 1 plus 1 plus 1). Additive composition also justifies another permission that is central to the subtraction algorithm: the permission to calculate by partition. In doing a calculation, it is permissible to divide the quantities being operated on into any convenient parts, operate on the parts, and accumulate partial results. This allows subtraction to proceed column-by-column.

Calculation by partitioning is, however, subject to several *constraints*. In the case of subtraction, these constraints specify that (a) each part of the subtrahend (the bottom number) must be subtracted from a part of the minuend (the top number); (b) each part of the subtrahend may be subtracted only once (thus, each subtrahend part may be "touched" only once); (c) all of the subtrahend parts can be removed sequentially from the same minuend part (thus, some minuend parts may be touched several times, and others may not be touched at all); and (d) in summing the partial results, any minuend part that has not been touched must be treated as if it were the result of a subtraction.[2]

In the course of calculating by partitioning, it may be convenient to recompose the parts. In the case of subtraction, such recomposing is done to avoid accumulating negative partial results. Thus, when the top number in a column is smaller than the bottom number in that column, one adds to the top number to make it larger. This is called *borrowing* or, in more modern school parlance, *regrouping*. Regrouping is permitted by the additive composition principle but is subject to an essential constraint: Addition in one column must be compensated by subtraction in another column, so that the total quantity in the top number is conserved. The constraint of conservation via compensation is necessary because the original goal of the algorithm is to find a difference. If either number was allowed to change in the course of calculation, the difference between the numbers would also change.

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[2] This is what children are taught in school under the name of "bringing down" the top number in a column when the bottom of the column is empty; it is equivalent to subtracting 0 from each minuend part that has not otherwise been touched.

This kind of analysis of algorithms as interacting permissions and constraints, each of which derives from basic features of the number system, allows us to give a new and more specific meaning to the idea of understanding rules and procedures. One understands a rule or procedure when one knows all constraints and permissions governing it. Greeno, Riley, and Gelman (1984) have shown that such analysis allows strong inferences about children's understanding of counting, even when the children are unable to verbalize explicitly their knowledge of constraints and permissions. Particularly strong inferences about understanding can be made when children construct variants of a standard procedure. In such cases, we can analyze the newly constructed procedures to see which constraints have been violated and which have been respected. If a constraint is violated, we can infer that the child either does not know the principle justifying the constraint or has failed to recognize its appropriateness to the procedure under construction. I will analyze here several examples of such invented procedures.

Partitioning and Recombining in Calculation

A particularly rich set of examples of principle-based informal mathematical reasoning comes from a longitudinal case study we conducted of a single child's invented arithmetic (Resnick, 1986). We began to study Pitt's mathematical knowledge when he was 7 years and 5 months old.[3] At the time, he had just finished first grade. As will become clear, Pitt was unusually flexible in his arithmetic procedures. He enjoyed arithmetic and participated with great eagerness in our interviews. His value to us lay in the great variety of invented procedures that he used (because he was working somewhat ahead of his school instruction) and in the exceptionally articulate explanations and justifications he gave for what he was doing.

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[3] This work was conducted in collaboration with Mary Means.

A first sign of Pitt's command of additive composition came when he was faced with the task of counting a large disorderly pile of monopoly money. He was 7 years and 5 months old at the time. After being shown the pile and told to count it in any way he wanted, he was asked to say how he planned to do it. He responded,

> *Well, I'm going to get the most first, then the second most. Like I'm going to get the five hundreds first . . . take out all the 500's and count those. Then take out all the 100's and count those. And add those two up. Then I'm going to go to the 50's, and the 20's, then the 5's and the ones. . . .*

Here we see that Pitt knew that it was permissible to partition the task of counting, to count some portion of the money, then another portion, then combine his partial counts. This simple and primitive permission derives from the composite nature of numbers.

Pitt worked for awhile with his largest-first strategy, but lost track of his counts somewhere in the 7000's. To help him remember (he was explicit about the reason), he did two things: he wrote down some partial amounts, and he began to group the bills into round amounts before counting. He said,

> *I'm going to put all the 500's in 5000's; then I'll add up all the 5000's.* [Pitt put the $500 bills together in groups of ten and counted them by five thousands; Pitt put

> $50 bills together in pairs and counted them by hundreds]. . . *I'm making up all the 20's and trying to make 100 so I can just have 7200. . . . I'm doing it differently 'cause I have two twenties and a ten, makes fifty. Then I have fifty here. And adding that up makes one hundred. So now . . . 7300. . . . Well, 4 twenties. Now I'm going to get 2 tens since there's no more twenties, and that'll make one hundred . . . 7400.*

In this sequence, Pitt showed great flexibility in composing his round groupings. This is evidence of his understanding that a number can be composed in many different ways and still be the same number. At the same age, Pitt's invented addition and subtraction procedures demonstrated additional aspects of his understanding of the compositional structure of numbers. In the following example, he is adding 152 and 149, which have been stated to him orally:

> *I would have the two 100's, which equals 200. Then I would have 50 and the 40, which equals 90. So I have 290. Then plus the 9 from the 49, and the 2 from the 52 equals 11. And then I add the 90 plus the 11 . . . equals 102.* 102? *101. So I put 200 and the 101, which equals 301.*

Here we see Pitt using the key permission, derived from the additive composition of number principle, of computation by partitioning. In this case, Pitt broke each of the numbers into three components. The three components were not a random choice but responded to the decimal structure of our system for naming and notating number. He then added convenient components and accumulated everything at the end.

The unorthodoxy of Pitt's method, with respect to what would be taught in school a year or so later, can best be conveyed by the written work he did when asked to explain his method of adding 60 and 35. Pitt's writing appears in Figure 1. His accompanying verbal description was:

> *I would take away the 5 from the 35. Then I'd add the 60 and the 30, which equals 90. Then I'd bring back the 5 and put it on the 90, and it equals 95.*

Insert Figure 1 here

Here Pitt temporarily removed a component from one of the numbers in order to allow him to use a known "number fact" (60 + 30 = 90) but then brought back the removed component at the end. Thus, he knew that it is permissible to change a number in the course of calculation, as long as a compensating change is made at another point, so that the total quantity is preserved.

Other examples of invented procedures that depend on the permission to decompose numbers, and of the accompanying compensation constraint, come from the work of children in a mathematics program in which there is no teaching of standard algorithms. Instead, children invent and discuss multiple solutions to problems (Resnick, Bill, & Lesgold, in press). Figure 2 shows several solutions developed by children for a single story problem. The notations are copied from notebooks kept by the children to record the procedures proposed and explained by each of several working groups in the classroom. The class first developed an estimated answer--not enough barrettes. They were then asked by the teacher exactly how many more barrettes would be needed. Additive composition principles come into play in the different ways in which the children interpreted the problem. Group 1 interpreted the problem as a missing addend addition problem, 36 + _ = 95. Groups 2 and 4 interpreted it as a subtraction problem, 95 - 36 = _, in Group 2's notation. The children's ability to accept the two interpretations as equivalent depends on their appreciation of the principle of complementarity of addition and subtraction.

Insert Figure 2 here

Additive composition principles also came into play in the way children performed the computations. Group 1 first found the number of barrettes by repeated addition, then checked this result by a decomposition of 4 X 9 into (2 X 9) + (2 X 9). This decomposition depends on an implicit understanding of the principle of distribution. We also see the principle of compensation at work in Group 1's solution. They solved their missing addend problem using an estimation procedure in which they tried adding a round amount, 60, to 36. This yielded 96, 1 more than the 95 specified in the equation they had written. They therefore subtracted 1 from 60, yielding their final answer of 59. Group 2 showed an even more sophisticated understanding of decomposition, for they included a negative partial amount in their computation. Using a place-value decomposition permission, they avoided regrouping by noting -1 as the result of 5 - 6. They then combined the two partial results, 60 and -1, to yield the answer of 59.

Other examples of decomposition methods of doing computation, all involving procedures quite different from school-taught algorithms, come from research on minimally schooled children and adults carrying out the arithmetic computations associated with their everyday work. For example, Schliemann and Acioly (1989) studied the methods used by lottery ticket vendors in Northeastern Brazil. They found that the vendors often calculated amounts owed for particular bets by methods of partitioning and grouping. Partitioning methods were similar to those used by Group 2 in Figure 2, except that the partitions were not always based on place value decomposition. That is, the quantities involved were partitioned into subtotals, necessary operations were performed on these subtotals and the partial results were then reunited. For example, a bookie with 5 years of school experience given the price of a bet of 1 cruzeiro at 50 cents on each of 240 different digits as 360 cruzeiros. Asked how he had calculated the total, his answer was:

> *Because I know that [the bet] for 1 cruzeiro . . . makes 240. Since it was 1 and 50, and 50 is half of 1, this makes 120 more. Then you add. This makes 360.*

Grouping methods observed were more like those of Group 4 in Figure 2. That is, quantities were operated on iteratively until the desired result was reached. For example, here is a bookie's calculation of the price of a two part bet:

> *On the thousands [one part of the bet] you have 16, because 4, 8, 12, 16. On the hundred [the second part of the bet] you have [pause] 28 + 28 makes 56, and 56 plus 56 makes 112. 112 plus 16 makes 128.*

Similar partitioning and grouping procedures have been reported by Carraher (1990) for construction site workers and by Carraher, Carraher and Schliemann (1985) for street vendors.

Commutativity and Associativity

Many of the partitioning solutions discussed above involved implicit application of principles of commutativity or associativity of addition, principles that permit numbers to be combined in any order. It is useful, however, to examine more systematic evidence of how children understand this principle. Again we can turn to Pitt, our enthusiastic longitudinal participant, for an explicit verbalization of the principle. For Pitt, commutativity and associativity of addition were permissions, rooted in additive composition, that seemed self-evident. When (at 7 years, 7 months) he was asked to add 45 and 11, and then immediately afterward 11 and 45, he simply repeated his first answer and said, "They're the same numbers, so they have to equal the same thing." Commutativity, in other words, was not a special law for Pitt. It derived from the same principle that allowed him to partition addition problems in different ways but still get the same answer, because, in his words, he "used all the numbers . . . that were in the adding problem but not in the same order."

Much younger children than Pitt, preschoolers who still perform addition by counting, show evidence of implicitly understanding the permission to commute when adding. When given addition problems to solve, preschoolers and kindergarten children know that they should combine the objects in two sets and then count the combined sets. If problems are given verbally, they will first count out the number of objects in each of the sets separately and then recount the combined set. Fuson (this volume) and Carpenter and Moser (1984) provide a nice description of this process, which is called *count-all*. A more sophisticated procedure, *count-on*, starts with the number in the first set and counts on from there, with the number of counts equal to the number in the second set. In this procedure, 3 and 5 would be added by saying, "three . . . four, five, six, seven, eight." Counting on requires some way of mentally keeping track of how many counts have been made. The number of counts needed and, thus, the difficulty of keeping track of counts, can be reduced by applying the commutativity permission. Using this procedure, known as the MIN procedure because the number of counts is equal to the smaller of the two addends, 3 and 5 would be added by saying, "five. . . six, seven, eight." Groen and Resnick (1977) showed that children as young as kindergarten invented the MIN procedure, thus providing evidence that children implicitly appreciate the commutativity permission.

Resnick and Omanson (1987) reported a sophisticated application by second and third graders of the associativity principle in the context of an invented procedure for mental addition. Using a mixture of reaction time and interview data, they showed that several children added problems such as 23 + 8, by decomposing 23, yielding (20 + 3) + 8 and then reconfiguring the problem to (20 + 8) + 3. Because (20 + 8) could be recombined to 28 very quickly on the basis of place value knowledge, this allowed them to apply a simple counting-on solution: "twenty-eight . . . twenty-nine, thirty, thirty-one."

Compensation and Equivalence

I have already mentioned the role of compensation as a constraint on the decompositions and recompositions used in calculation. But compensation plays a special role in developing concepts of equivalence, which are, of course, essential to algebraic thinking. Here again, Pitt gives us particularly articulate expressions of the kind of intuitive knowledge that is at work in many children's invented solutions. At 7 years, 7 months, he was asked how 23 + 41 (written vertically on paper) could be rewritten so that it would still equal 64. He first said 24 + 40 and then continued:

> *I'm going one less than 40 and this one more . . . 25 plus 39.* Tell me what you're doing now to get that. *I'm just having one go lower; take one away and put it on the other. . . . I'm taking the 3* [from 23] *away and making that 2, and putting it on the 41 to make it 42. Like that, I was going lower, lower, higher, higher.* Okay, you gave me three examples of how you could change the numbers. Now why do all those numbers equal the same amount? *Because this is taking some away from one number and putting it on the other number.* And that's okay to do? *Yes.* Why is that okay to do? *Why not?* Well, can you give me a reason? *No, anyone can do that. . . . Because you still have the same amount. You're keeping that but putting that on something else. . . . You're not just taking it away.*

Here Pitt demonstrated his understanding that one can think of the original numbers, 23 and 41, as composite parts of the larger number, 64, and that one is free to recompose the 64. Furthermore, he showed that he knew that one can conserve the whole by compensating a reduction in one of the parts with an increase in the other part. He also expressed verbally a critical constraint on recompositions: It is permissible to move part of a number to another number but not simply to "[take] it away." This is the heart of the compensation constraint on recompositions: To maintain equivalence, one must compensate changes made in one part of a number with equal changes in another part.

At an even younger age, another child we have studied extensively used his knowledge of compensation and equivalence to invent an efficient procedure for solving problems of the form, "If you have six marbles and I have four, what could we do to each have the same number?" (Resnick, 1986)[4]. There are three basic methods for solving this problem. In the simplest method, which we called *buy/sell* following David's (our case study child's) language, the larger set is diminished or the smaller increased by the difference between the two sets. In the second method, *share*, the objects in the two sets are (mentally) "put in the middle" and then each person takes half. In the third method, the most complex and the object of our microgenetic study, enough objects are transferred from the larger to the smaller set to make the two sets equivalent. This is called *transfer.*

David, first interviewed at 6 years, 2 months, had full mastery of buy/sell and share, even when the problems involved rather large numbers. Transfer was demonstrated to him by the interviewer in a second interview a few weeks later. He clearly understood the goal of direct transfer. Further, if a number to be transferred was proposed to him, he could calculate its consequences for both sets and decide whether or not it met the goal of equalizing the sets. However, he could not himself generate the number of objects to be transferred. David's early ability to grasp the *goal* of directly transferring objects from one set to another, together with his ability to evaluate the effects of a proposed transfer, allows us to attribute to him a basic schema for *transfer* that might be represented graphically as in Figure 3. This schema includes knowledge that one can think of the two initial sets as parts of a whole superset and that one can repartition the superset so that each part contains the same number. It also includes knowledge that one can move elements of the larger set into the smaller set and that this movement will make the large set smaller and the small set larger. Thus, although David could not yet do all of the numerical arithmetic necessary, he understood in what we have come to call *protoquantitative* terms (a more complete discussion will follow) both the permission to repartition and the constraint of maintaining equivalence.

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[4] This work was done in collaboration with Terry Green.

Insert Figure 3 here

David's gradual acquisition of the full transfer strategy over a ten-month period constituted the main focus of the case study. Initially David could only guess at what number to move from the large to the small set. However, his guesses never exceeded the difference between the two sets. We can conclude, therefore, that David understood that he must move the difference amount or some part of it over to the smaller set. During the 10 months, David gradually quantified his transfer schema, eventually arriving at a systematic rule of transferring half the difference between the two sets.

Many other examples of children's appreciation of the compensation/equivalence constraint have appeared in our studies. I have already mentioned its role in the estimation-based solution of Group 2 in Figure 2 and in the justification for regrouping in conventional multidigit addition and subtraction algorithms. In an instructional experiment aimed at helping children overcome the tendency to interpret the rules of written subtraction entirely in terms of rules for manipulating symbols, without reference to the quantities exchanged in the course of borrowing, Resnick and Omanson (1987) focused children's attention on the quantities removed from or added to columns by decrement and increment marks. After such instruction, many children constructed explanations of the subtraction algorithm in terms of compensating additions to one column and subtractions from another. For example, one child described what she had done when she produced the notation, ${}^{2}\cancel{3}\ {}^{9}\cancel{0}\ {}^{9}\cancel{0}\ {}^{1}2$, as taking 1000 from the 3 and distributing it to the other columns: 900 to the hundreds column, and 100 to the tens plus the ones columns, broken up into 90 and 10. Because the instruction never directly discussed parts and wholes or the idea of conserving a quantity while redistributing its parts, it seems appropriate to conclude that the children already had a compensation/equivalence principle available that they used to interpret the scratch marks of written borrowing as soon as their attention was focused on the quantities to which the marks referred.

Perhaps the most systematic study to date of children's understanding of compensation and equivalence constraints is that of Putnam, deBettencourt and Leinhardt (1990). They asked third grade students to watch puppets demonstrate "derived fact strategies" for addition and subtraction, to complete the calculations, and then to justify them. Derived fact strategies transform presented problems into problems that are easy to solve, because they use well known addition and subtraction facts. The transformations in one number require compensating transformations in another. For example, 3 + 4 can be transformed to (3 + 3) + 1 to allow use of the familiar doubles fact. In the transformation, 1 is subtracted from the 4 but then added back to the result. In another case, 7 + 9 can be transformed to 8 + 8. In this case, 1 is subtracted from 9 and added to 7 in order to preserve the whole. Table 1 shows the full set of transformation and compensation rules that could be used to justify the derived fact strategies they studied. They found that 50 to 60% of the third graders interviewed could explicitly justify addition-derived fact strategies with verbalizations expressing these rules. Many more could complete the strategies and give partial explanations, perhaps indicating a more implicit appreciation of the rules. Subtraction-derived fact strategies were much more difficult for the children to justify, with only 10 to 20% providing complete, explicit justifications.

Insert Table 1 here

Additive Inverse: Complementarity of Addition and Subtraction

The examples already considered include several cases in which the principles of the complementarity of addition and subtraction were applied. For example, the second grade class whose work is shown in Figure 2 recognized as equivalent two different interpretations of the same story problem: one an unknown addend addition interpretation; the other a subtraction interpretation. More systematic evidence of children's understanding of the additive inverse principle comes from research on the mental counting procedures young children use to solve problems presented as numerical subtraction problems. Several studies (e.g., Svenson & Hedenborg, 1979; Woods, Resnick, & Groen, 1975) have shown that, starting at about age 7, children figure out the answer to these problems by either counting down from the larger number *or* counting up from the smaller number, whichever requires the fewest counts. Thus, children do the problem, 9 - 2, by counting down: "nine . . . eight, seven . . . the answer is seven"; but they will do the problem, 9 - 7, by counting up: "seven . . . eight, nine . . . the answer is two." This *choice* procedure, as it has become known, has been inferred from a combination of interview protocols and the pattern of reaction times for subtraction problems with different numbers.

Applying the choice procedure means that children must convert some subtraction problems into addend-unknown addition problems, for example, 9 - 7 = _ converts to 7 + _ = 9. Children's willingness to treat these two problems as equivalent means that, at least implicitly, they understand the principle of complementarity. This principle, in turn, depends on an additive composition interpretation of the problem in which 9 is understood to be a whole that is decomposed into two parts, one of which is 7. In this interpretation, the problem becomes finding the other part, and it does not matter whether one subtracts a known part from the whole to find it or starts with the part and determines how much more is needed to make up the whole.

Multiplication and Division Interpreted Additively

One further point is needed to complete the story of the primacy of additive composition in children's early understanding of arithmetic. This is the tendency of children to prefer additive composition interpretations even of situations that adults might understand in terms of multiplicative relations between numbers. Again, Pitt provides a particularly compelling example. Remember that he found commutativity of addition to be so self-evident that he found it somewhat difficult to provide a justification. Commutativity of *multiplication*, however, was not so self-evident for him; it required an explanation. Although he freely used the commutativity principle in his multiplication calculations, he did not claim that multiplier and multiplicand could be inverted just because they are the same numbers. Instead, he justified the commutativity of multiplication in terms of *additive* compositions:

> What's two times three? *Six.* How did you get that? *Well, two threes . . . one three is three, one more equals six.* Okay, what's three times two? *Six.* Anything interesting about that? *They each equal six, and they're different numbers. . . . I'll tell you why that happens. . . . Two has more ways; well, it has more adds . . . like two has more twos, but it's a lower number. Three has less threes but it's a higher number. . . .* Alright, when you multiply three times two, how many adds are there? *Three . . . And in the other one there's two. But the two--that's two threes--but the other one is three twos, 'cause twos are littler than threes but two has more . . . more adds, and then the three has less adds but it's a higher number.*

Ginsburg (1977) provides a number of examples of children's invented solutions to problems involving multiplication, division, and fractions. In virtually every case, children relied on knowledge of *additive* properties of number to find solutions, as they did in the following:

> 6 x 8. Okay why don't you write that down? *6 x 8 is . . .* [wrote down 6 + 6 + 6 + 6 + 6 + 6 + 6 + 6.] O.K. Do you expect to get the same answer from this problem [referring to the 6 + 6, etc.] as this problem [referring to the 6 x 8]? *48.*

> O.K. How did you get 48? *Well I did four sixes . . . I mean I added them together. And then I added; the answer was 24, so I added them together, and that was 48.* [wrote down 24 + 24 = 48.] (Ginsburg, 1977, p. 99)

> How many are half of these? *15.* Can you prove that? Can you convince me it's true? *I don't think so.* O.K. How many is a quarter of these blocks? One fourth? That might be too tough. I won't ask you that. *Wait . . . 5 and 1/2 . . . 7 and l/2.* Very good. Now tell me how you got 7 and l/2. *I did 8 and 8 is 16, and so that's just one more, so it would be 7-1/2.* (Ginsburg, 1977, p. 102.)

A further example of the tendency to prefer additive over multiplicative solutions comes from a study of children who work as street vendors in Recife, Brazil:

> *How much is one coconut?* 35. I'd like ten. How much is that? [Pause] *Three will be 105; with three more, that will be 210.* [Pause] *I need four more. That is . . .* [Pause] *315. . .* . . I think it is 350. (Carraher, Carraher, & Schliemann, 1985, p. 23).

Evidence of Systematic Difficulty in Learning School Mathematics

The evidence just assembled seems to fly in the face of common experience with the difficulties of mathematics learning. It suggests that mathematical ideas based on additive composition are accessible to children and may be universally mastered, even by people with little or no schooling. Yet many children have a great deal of difficulty learning school mathematics.

The phenomenon of math anxiety--extreme lack of confidence in one's ability to cope with mathematics--is familiar in virtually all highly educated societies. Many who proceed at accepted rates in standard mathematics instruction have little taste for it and seem unable to use their knowledge flexibly and creatively. Why should strong and reliable intuitions of the kind documented for young children and unschooled people not be sustained in school mathematics learning?

I consider two important hypotheses, which are not mutually exclusive. One is that the focus in school on formal symbol manipulation discourages children from bringing their developed intuitions to bear on school mathematics learning. I call this the *syntax-semantics* hypothesis. The second hypothesis is that there is an epistemological discontinuity between informal mathematics rooted in everyday behavior and the kind of mathematical reasoning that is sought in school. That is, because formal school mathematics calls for reasoning about abstract entities--numbers, operators, relationships--that cannot be directly experienced in the physical world, mathematical competence "on the streets" may do little to prepare children for formal mathematics participation. I call this the *abstract entities* hypothesis.

The Syntax-Semantics Hypothesis

A recurrent finding in studies of arithmetic learning is that children who are having difficulty with arithmetic often use systematic routines that produce wrong answers. This observation has been made repeatedly over the years by researchers concerned with mathematics education, and several studies have attempted to describe the most common errors. Systematic procedural errors have been documented for many topics in school arithmetic and for algebra. Investigation of *buggy algorithms* and *malrules* by cognitive scientists has yielded automated diagnostic programs capable of reliably detecting the particular errorful algorithms used by a child on the basis of response to a very small but carefully selected set of problems. Formal theories of the reasoning processes by which children invent incorrect procedures have also been constructed. All of these studies point to the conclusion that systematic errors result from applying intelligent forms of reasoning, such as generate-and-test problem solving heuristics, to a knowledge base devoid of representations of quantity and filled only with rules for operating on symbols. Because quantities are not represented, reasoners often cannot recognize and do not apply mathematical principles derived from knowledge of the additive composition of quantities when doing school mathematics.

Some examples of buggy procedures and their analyses will help to make clear what is meant. The domain that has received the most careful analysis is written subtraction with borrowing (Brown & VanLehn, 1982; Burton, 1982; VanLehn 1990). A finite number of bugs, which in various combinations make up several dozen buggy algorithms, has been identified for subtraction. Here are two of the most common bugs:

Borrow-From-Zero. When borrowing from a column whose top digit is 0, the student writes 9 but does not continue borrowing from the column to the left of the 0.

$$\begin{array}{rr} 6\,{}^{9}\cancel{0}{}_{1}2 & \qquad 8\,{}^{9}\cancel{0}{}_{1}2 \\ \underline{-437} & \qquad \underline{-396} \\ 265 & \qquad 506 \end{array}$$

Borrow-From-Zero. When the student needs to borrow from a column whose top digit is 0, he skips that column and borrows from the next one. (This bug requires a special "rule" for subtracting from 0: either 0 - N = N or 0 - N = 0.)

$$\begin{array}{rr} {}^{5}\cancel{6}02 & \qquad {}^{7}\cancel{8}0{}_{1}4 \\ \underline{-327} & \qquad \underline{-456} \\ 225 & \qquad 308 \end{array}$$

These examples show that the results of buggy calculations tend to "look right." They also tend to obey a large number of the important rules for manipulating symbols in written calculation: There is only one digit per column; all columns are filled; there are increment marks in some columns with (usually) decrements to their left, and so forth. The buggy algorithms seem to be orderly and reasonable responses to problem situations. On the other hand, if we look beyond the symbol manipulation rules to what the symbols represent, the buggy algorithms look much less sensible. Each of the bugs violates fundamental mathematical principles (Resnick, 1982).

For example, *Borrow-From-Zero* looks reasonable at first glance, because it respects the requirement that in a borrow there must be a crossed-out and rewritten numeral to the left of the column that is incremented. It also respects the surface rules for the special case of zero, where the rewritten number is always 9. However, the bug violates the fundamental principle that the total quantity in the top number must be conserved during a borrow. Interpreted semantically (that is, in terms of quantities rather than simply as manipulations on symbols) a total of 100 has been added: 10 to the tens column and 90 to the hundreds column. Similarly, *Borrow-Across-Zero* respects the syntactic rules for symbol manipulation requiring that a small 1 be written in the column that is incremented and that a nonzero column to the left be decremented. Like the previous bug, however, it violates the principle of conservation of the top quantity. In this case 100 is removed from the hundreds column, but only 10 is returned to the units column.

In these two bugs, as in all of the others observed for subtraction, constraints imposed by the quantitative meaning of the symbols (the semantics) are dropped, but constraints derived from the rules of symbol manipulation (the syntax) are retained. The same separation of syntax from its underlying semantics seems to be the case for systematic errors in other parts of mathematics, although the particular forms, of course, vary. For example, Matz (25) has argued that many algebra errors can be attributed to a process of extrapolating new rules from "prototype rules." An example appears in Figure 4. The initial rule is the distribution law as it is typically taught in beginners' algebra courses. From this correct rule a prototype is created by generalizing over the operator signs. From this prototype, new, incorrect distribution rules can be constructed by substituting specific operations for the operator placeholders in the prototype. As for the buggy subtraction rules, there is no representation of the quantities and relationships involved in the algebra expression or its transformation. Instead, the malrule results from deformation of rules of symbol manipulation.

Insert Figure 4 here

Resnick, Cauzinille-Marmeche and Mathieu (1987) explicitly pitted syntactic versus semantic interpretations of algebra expressions in a study in which French children between 11 and 14 years of age made judgments about the equivalence of algebraic and numerical expressions. For example, several types of items provided opportunities for children to express knowledge of a principle they termed *composition of quantity inside parentheses*. This principle expresses the fact that the two terms inside parentheses in an expression such as $a - (b + c)$ are the parts of a single whole quantity and that this whole quantity is to be subtracted from the starting quantity, a. This principle can be used to explain the sign-change rules for removing parentheses. We asked children, whether pairs of expressions, such as $a - (b + c)/(a - b) + c$, $a + (b - c)/(a + b) - c$, and $(a - b) - c/a - (b - c)$ were equivalent or not and why. Children predominantly used rules they had learned in school, or deformations of these rules, to make the judgments. One common error was to focus preemptively on parentheses, claiming that if the material inside parentheses was different, two expressions could not be the same. This led to judgments such as $a + (b - c) \neq (a + b) - c$. This malrule probably results from an intrusion into the algebraic system of a rule for numerical expressions that calls for operating inside parentheses first.

A second common parentheses error seemed to derive from deformation of an algebraic transformation rule rather than an intrusion of a calculation rule. This malrule claimed that the placement of parentheses was irrelevant, as long as the letters and signs retained their positions. Thus a - (b - c) was judged equivalent to *(a - b) - c*. Some children justified this equivalence by calling on the formal rule of associativity, misapplying it to subtraction. Another purely formal error was to apply the law of commutativity to subparts of an expression that were not enclosed in parentheses and thus did not warrant being treated as a separate quantity. For example, $a - b + c$ and $a - c + b$ were judged equivalent because $b + c$ commutes to $c + b$.

Another domain in which systematic errors have been documented and analyzed is decimal fractions (Hiebert & Wearne, 1985; Sackur-Grisvard & Leonard, 1985). Following up on that earlier work, Resnick et al. (1989) showed that the errorful rules for comparing decimal fractions could be classified into two basic categories. One class of errors applied a whole number rule in which rules for comparing whole numbers were incorrectly applied to the fractional part of a number. Children applying this rule would consistently judge as larger, the decimal number with more digits. Thus, they judged $4.63 > 4.8$, $0.36 > 0.5$ and $0.100 > 0.25$, giving reasons such as, "63 is bigger than 8." Children making the second class of errors consistently judged as larger, the decimal number with *less* digits, yielding judgments such as $4.45 > 4.4502$ and $2.35 > 2.305$. We called this the *fraction rule* error because it resulted from children's efforts to integrate knowledge about fractional parts and ordinary fraction notation with their incomplete knowledge of the decimal notation system. These children knew that if a number is divided into more parts, the parts are smaller, correct semantic knowledge about quantities. They also knew, another correct piece of semantic knowledge, that a number cut into thousandths has more parts than one cut into hundredths, which in turn has more parts than one cut into tenths. So they judged a number that had thousandths or ten thousandths in it to be smaller than one that had only hundredths.

The tendency to separate quantitative and symbolic representations seems to be a major stumbling block in school mathematics learning. When working with mathematical notation, one does not automatically think about the quantities and relationships that are referenced. What is more, school instruction probably tends to aggravate this tendency for the formal notation of mathematics to function independently of its referents. The focus in elementary school is on correct ways to perform procedures, a focus largely detached from reflection on the quantities and relationships to which symbolic expressions refer. This probably encourages children to attend to formal notations and rules for manipulating them without relating these rules to the semantics--that is, to the external referents--of the notations.

The pervasiveness of semantically unconstrained, syntactic ways of thinking about school mathematics and of the systematic errors that this way of thinking appears to induce seems to suggest that a major revision of school mathematics instruction in the direction of rooting it in the kinds of semantic knowledge that children seem to have when they first come to school would do much to limit errors and enhance mathematical understanding. This is a widely promoted idea among mathematics educators and psychologists. It is at the heart of many proposals to use manipulatives and graphic models in early mathematics teaching. However, there is some evidence that even very carefully crafted lessons that use manipulatives to establish the semantic principles underlying calculation procedures do not, by themselves, succeed in establishing a propensity for attending to the semantics of procedures. Resnick and Omanson (1987) used a *mapping* procedure to teach the semantics of multidigit subtraction to children who had used buggy procedures on two different tests. In mapping instruction, conducted in individual tutorials, children used Dienes blocks (which physically represent the value of numerals in the columns) to perform the various steps of exchange involved in subtraction with regrouping and recorded each step. In this way, the actions on the physical quantities--well constrained by the principles of additive composition--would, we thought, generate and thereby give meaning to the written notations of the algorithm. Despite the intensive personal instruction, however, only half the children taught learned the underlying semantics well enough to construct an explanation of why the algorithm worked and what the marks represented. More surprising, even children who did give evidence of good understanding of the semantics often reverted to their buggy calculation procedures once the instructional sessions were over.

This result does not, of course, suggest that there is no value in using manipulatives to teach the meaning of algorithms. Especially since our study involved children who had already established buggy patterns, it does not tell us what to expect from instruction that initially introduced computational procedures on the basis of manipulatives and other semantic aids. I think, however, that it does suggest that simple reliance on manipulatives to teach algorithms does not go far enough in capturing and building on children's informal mathematics knowledge. Instead, instruction that develops a fundamental *attitude* toward arithmetic as grounded in meaningful relationships is probably needed. This would require far more than occasional lessons using manipulatives or explaining an algorithm. Rather it would mean grounding the entire learning program in problems that treat numbers as representations of real-world quantities. This is what some successful early mathematics programs have done (see e.g., Carpenter, Fennema, & Peterson, 1987; Cobb et al., 1991; Resnick, Bill, & Lesgold, in press).

The Abstract Entities Hypothesis

At some point, however, we must expect an instructional program focused entirely on relations among physical quantities to founder. This is because of three features that distinguish mathematical knowledge from most other forms of knowledge. First, mathematics is concerned not just with physical quantities, but with abstract entities--numbers, operators, functions, etc.--that cannot be directly observed in the world. Second, mathematical knowledge is intimately linked to a specialized formal language that both imposes constraints on mathematical reasoning and confers extraordinary power. Third, the formal language of mathematics plays a dual role as *signifier* and *signified*, as both the instrument of reasoning and as the object of reasoning.

Mathematics as abstract knowledge. In all domains of knowledge, forming a concept requires abstractions that go beyond individual objects that can be denoted. The concept of a chair or a dog requires that children construct a representation that abstracts over the specific dogs or chairs they may encounter. In most domains, however, there are at least *members* of a class or concept that one can point to, and it is possible to reason about specific cases. This is true even for so-called abstract concepts such as freedom, beauty, illness, and the like. We can define such concepts inductively, at least in a loose sense, by collecting examples of them. Mathematics does not have this property. There are not, strictly speaking, denotatable objects in mathematics. For example, although one can point to a set of three things and to the written numeral *3*, these physical objects do not in themselves have the property of number. Number, as such, is a *conceptual* entity (cf. Greeno, 1983). This means that before people can reason in truly mathematical ways they must engage in a process of constructing the conceptual entities they will reason about.

Mathematics as formal knowledge. Although one can reason about some aspects of quantity without using any written notation, there are very strict limits on how much reasoning about number one can do without using formalisms. Even counting engages a formalism, for a standard set of labels (the count words) must be used in a standard sequence, and these labels must be paired with objects in accordance with strict constraints (one label to one object, each used once and only once, in a fixed order). The dependence of mathematical reasoning on formalisms becomes more marked as one proceeds to more complex levels of mathematical development. Many concepts in arithmetic, for example, division by a fraction or subtraction of a negative number, can only be understood within a system of formal relationships. Efforts to explain these concepts with respect to physical quantities are cumbersome and limited in their application. The central role of formalism in mathematics becomes particularly evident when one considers algebra, where the formalism allows one to reason about operations on numbers and relations between numbers without reference to any particular numbers.

The dual role of mathematical language. Throughout mathematics, the terms and expressions in the formal notation have both referential and formal functions. As referential symbols, they refer to objects or cognitive entities external to the formalism. As formal symbols, they are elements in a system that obeys rules of its own, and they can function without continuous reference to the mathematical objects they name. For example, when one applies the count words in sequence to individual objects in the course of counting, one is using the count words as purely symbolic tokens in a formally constrained procedure. The words do not refer to anything; they just keep the procedure running appropriately. However, when the same words are used to name the cardinality of the whole set that has been counted, they are names that have a referent, albeit a more abstract referent than many of the names in natural languages.

The dual role of mathematical symbols is particularly obvious and complicated in the case of algebra. One great power of algebra is that it allows extensive manipulation of relationships among variables within a completely reliable system that does not require continuous attention to the referential meaning of the intermediate expressions generated. The fact that the algebra system can be "run on its own," so to speak, is surely a factor in favor of its efficiency. Potentially unbearable demands on processing capacity would be placed on individuals who tried to reason through some of the complex problems for which algebra is used if, at each step, they were considering physical, situational, or specific quantitative referents for the transformations they produced. But algebra is not only a device for reducing capacity demands. Its very abstraction away from the situations, quantities, and relationships that are its referential meaning is part of what permits certain mathematical deductions to be made.

These characteristics of mathematical thinking mean that to think mathematically it is necessary to go beyond the kinds of intuitions that can be related directly to physical embodiments of quantity. Mathematical entities must be constructed and fluency in reasoning about those entities developed. This process may be a source of difficulty and at least temporary blockage in mathematics learning. It may be an example in mathematics learning of the kind of *epistemological obstacle* that Bachelard (1980) directed our attention to in science learning. This suggests that special attention may need to be paid in mathematics education to helping students construct and use the mental entities that constitute mathematical concepts.

Mathematical Entities in Elementary School Children's Reasoning

Some of our research has examined the beginnings of reasoning about mathematical entities. Our studies include examples of children at surprisingly young ages who seem to have entered the formal system and are able to reason about relations among abstract entities such as numbers. But they also reveal great difficulties in such reasoning.

Negative numbers. Considerable attention has been given to trying to develop pedagogical models to help children understand negative numbers (Davis, 1979; Janvier, 1985; Leonard & Sackur-Grisvard, 1987; Murray, 1985). Two basic classes of models have been developed: those that treat negative numbers as a special class of quantities, mirroring the behavior of ordinary quantities but sitting in a special "cancellation" relation to them; and those that treat negative numbers as elements in a formal system defined essentially by a number line with numbers moving outward in both directions from zero. Our research and that of others shows that children think most easily about negative numbers as quantities. That is, children and unschooled adults reason quite easily about debts or, for those living in cold climates, below zero temperatures. As young as 7 years or so, they understand that debts are created when payments are due that will more than use up an individual's current monetary resources; that debts can cumulate, essentially by addition "on the other side of zero;" that earning money (an action that would seem to imply addition when in the domain of positive quantities) reduces debts and can eventually--by crossing zero--eliminate debts and even create assets (Mukhopadhay, Resnick, & Schauble, 1990). Children can reason successfully about all of these aspects of negative quantities when these quantities form part of a meaningful story about a character with whose financial problems they identify, even when they are nearly totally incapable of solving simple written addition and subtraction problems involving negative numbers.

Analyzing other "street math" types of reasoning about quantities that the mathematically sophisticated represent with negatively signed numbers, Carraher (1990) argues that this form of reasoning, although practically powerful, in fact avoids the need for building a truly mathematical conception of signed numbers. An analysis by Peled (1989) implicitly seconds this claim. She lays out four levels of understanding of addition and subtraction of signed numbers. Her final level is essentially a formal one. It requires a number line representation of numbers qua numbers--i.e, numbers defined in relation to one another rather than as measures of physical quantities. It also requires a definition of the operations of addition and subtraction in terms of arbitrary directions of movement on this non-referential number line.

In an intermediate stage, when addition always means moving to the right and subtraction always means moving to the left, it is still possible to preserve at least a distant quantity referential meaning for these operations, for subtraction always means moving toward a quantitatively "lesser" quantity and addition always means moving toward a quantitatively "greater" quantity on the line. But in Peled's final stage, even these ordinary meanings of addition and subtraction and of the signs on the numbers are lost. A negative sign now means "flip" or "turn around"; addition means move in the direction you (or the arrow on the line) is facing; and subtraction means move backwards (i.e., in the direction opposite the one you are facing). All of these definitions are perfectly meaningful and "semantic" within the formal system, but to understand them requires accepting the formal system as a self-contained system of relationships with the semantics residing in the relationships among the elements in the system rather than in quantities external to the formal system. In our studies of elementary school children's understanding of negative numbers, we have encountered children as young as third grade who show an intermediate understanding of negative numbers as points on a number line (Peled, Mukhopadhay, & Resnick, 1988). But prior to formal instruction, we found none with a full formal understanding. Furthermore, many adults continue to be baffled by subtraction of a negative number, which is easily explainable only in the terms of a formal system.

Infinity. Although it is not normally an official part of the school curriculum, children are known to become interested in the concept of infinity at a surprisingly early age (Gelman & Evans). As young as 6 or 7, some children recognize that there is no largest number because it is always possible to add one more. This seems to depend on an appreciation of the formal numeration system, for the Gelman and Evans data also showed that the children who admit that *one more* can always be added are those who have fully mastered the recursion principle for count words--that is, they understand that when one reaches a boundary (such as 999) one can add a new token ("thousand" in this case) and then cycle through the entire sequence again. In cooperation with Gelman, we conducted a study that used the concept of infinity to further explore children's appreciation of numbers as elements in a strictly formal system (Bee et al., in preparation). We especially wanted to know when children began to recognize the dual function of mathematical notations (as signified and signifier) and the language they used to explain this idea when they did recognize it.

The children in the study were interviewed individually. The portion of the interview relevant to the present question proceeded in the following manner. If a child agreed that there was no largest number (for the ordinary, counting-by-ones sequence of numbers), she was then shown a string of even numbers (2, 4, 6, 8 . . .) and asked what the highest number in that string was. Similarly, she was asked what the highest number in the tens counting string (10, 20, 30 . . .) was. Most children who were firmly convinced that there was no largest ordinary number also concluded that the same thing must be true for the other number strings.[5] These children were then asked whether two number sequences (e.g., units and evens) had the same "number of numbers." If a child said that a sequence (usually the units sequence) had more numbers, she was shown that the numbers in the two sequences could be made to "hold hands" (as shown in Figure 5a) and asked whether this kind of pairing could go on forever. If a child said that two number strings were equivalent (either before or after the hand-holding discussion), she was shown a different pairing, where the numbers were matched by *values* (as in Figure 5b) and was invited to discuss the implications.

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[5] The interview included several countersuggestions by the experimenter and it is therefore possible to distinguish quite reliably between children who strongly believed there was no largest number and those who were not certain or were responding to cues from the experimenter as to what answers were desired.

Insert Figure 5 here

Several classes of responses to the equivalence questions can be distinguished. Some children (most of the younger ones) simply rejected the possibility that there could be just as many even numbers or tens numbers as regular numbers--even though they had observed and agreed to the possibility of the numbers holding hands forever. These children were responding to the number strings as signifiers, symbols referring to cardinalities. On this interpretation there are missing numbers in the twos and tens strings, and if numbers are missing then there must be fewer than in the complete set of numbers (the units string). Most sixth and eighth graders, however, recognized that the equivalence question posed a dilemma. They could follow and even accept the hand-holding, one-to-one correspondence argument, but they saw gaps in the twos and tens number strings as well. In the face of this dilemma, many children followed the interviewer's suggestions on both sides of the arguments, and shifted their positions several times in the course of the interview. Some resolutely focused on the numerical symbols themselves without reference to the cardinalities they referred to. That is, they focused on the symbols as signified rather than signifier. Here is an example of such a child:

> *There's the same amount.* Of counting numbers and even numbers? *Yeah.* How do you see that? *Because there's a match with every one of those. There's a two for a one, and a four for a two.* Before you said there was more of these, right? *Yeah.* What changed you mind? *When I see they all have a match so there must be the same.*

This child gave a "correct" answer to the equivalence question but had not really resolved the dilemma. It is by focusing on only one aspect of the number strings that she was able to respond. A few children, however, showed us in their responses what a full and conscious resolution of the dilemma would sound like in children's terms. They explained that both interpretations of the number strings were possible, and that one could arrive at either answer (the same number of evens as ordinary numbers, or fewer evens than ordinary numbers) depending on which one chose. Here is an example from one of the protocols:

> Why do you say there are just as many? . . . *Because you cannot . . . you could connect the. . . . Like the first even number would be two . . . first counting number would be one. They would be even, as they're both the first number. Then, the second.* I see, so you make . . . *You've got the same value of the number,* [on the hand holding model of the Figure, points to a number in the units then to the same number in the evens string] *but there's still the same number as. . . . You see this little symbol on your paper. There's the same number of little symbols.*

A Theory of Layers of Mathematical Knowledge

These considerations set the stage for a theory of "layers" of mathematical knowledge that can account for the passage from intuitive understanding rooted in knowledge of physical quantities to the ability to reason about mathematical entities. This theory is developed in detail in Resnick and Greeno (1990). The layers are distinguished by the kinds of entities that are recognized and reasoned about. Four layers are summarized in Table 2.

Insert Table 2 here

The most elementary layer is the *mathematics of protoquantities.* Protoquantitative reasoning is about amounts of physical material. Comparisons of amounts are made and inferences can be drawn about the effects of various changes and transformations on the amounts. But no numerical quantification is involved. The language of the protoquantitative layer of mathematical thinking is a language of descriptive and comparative terms applied directly to the physical objects or amounts: a *big* doll, *many* eggs, *more* milk. In the mathematics of protoquantities, operations are actions that can be performed directly on physical objects or material: increasing and decreasing, combining and separating, comparing, ordering, pairing. The simplest form of protoquantitative reasoning is direct perceptual comparison of objects or sets of different sizes. This permits recognizing the larger of a pair of objects, for example, something infants of three months are known to be capable of (Fantz & Fagan, 1973). More advanced protoquantitative reasoning works on a mental representation of amounts of material, and allows children to reason about the results of imagined increases and decreases in amounts of material. Thus, protoquantitative reasoners can say that there *will be* more apples after mother gives each child some additional ones, o· that some mice *must have been removed* if there are now less than before, without being able to look simultaneously at the objects in their *before* and *after* states. Similarly, mental combining and separating operations permit children to reason protoquantitatively about the relations between parts and whole--for example, more fruits in the bowl than either apples or oranges.

In the next layer, the *mathematics of quantities*, reasoning is about numerically quantified amounts of material. Numbers are used as measures: *4* dolls, *3 feet* of board, *7 pounds* of potatoes. In the mathematics of quantities numbers function like adjectives; they describe a property (the measured amount) of a physical quantity. The numbers take their meaning from the physical material they describe and refer to. Terms from formal mathematics, such as *add divide*, may enter the vocabulary, but their reference is to action on physical material. Operations in the mathematics of quantities are actions on measured amounts of material. For this reason there are several different kinds of addition, subtraction, multiplication and division. Addition can mean *combining* 4 apples and 3 apples or *increasing by 5* a set of 20 marbles. Subtraction can mean *taking away 20* potatoes from a bin containing 50 potatoes or *partitioning* a set of 15 cakes into 5 for the family and "the rest" to give to friends. Multiplication and division have even more possible referential meanings in the mathematics of quantities, such as *combining* 5 sets of 3 books each, *enlarging by a factor of 3* a 10-inch strip, *sharing* 20 cookies equally among 4 children. Always, in the mathematics of quantities, the reasoning is about specific quantities of material and actions on those quantities; the numbers function as descriptors of the quantities.

It is only in the next layer, the *mathematics of numbers*, that numbers acquire a meaning of their own. Numbers shift at this layer from functioning as adjectives to functioning as nouns. That is, the numbers themselves become conceptual entities that can be manipulated and acted upon. One can now add *3* and *4* (not 3 apples and 4 apples) or multiply *3* (not 3 books). In the mathematics of number, numbers *have* properties rather than being properties *of* physical material. The properties of numbers are defined in terms of other numbers. Numbers have magnitudes in relation to one another: for example, 12 is *8 more than* 4; it is *3 times* 4; it is *1/3 of* 36. Numbers are also compositions of other numbers: thus, 12 is *8 + 4, 7 + 5, 6 + 6*, etc. Operations in the mathematics of number are actions taken on numbers, resulting in changes in those numbers. Thus 12 can be changed to 4 by *subtracting 8* or, alternatively by *dividing by* 3. It can be changed to 36 by *multiplying by 3* or by *adding 24*. The numbers being compared, composed and changed in these examples are purely *conceptual* entities. Their meaning derives entirely from their relations to one another and their place within a system of numbers. Physical material need not be imagined.

It is not the fact of doing arithmetic mentally that distinguishes the mathematics of numbers from the mathematics of quantity. The mathematics of quantity, like the mathematics of numbers, can be done mentally--in one's head. One can think of splitting 12 apples into two collections, of 8 apples and 4 apples, say, or enlarging a 12 inch square photo to one that is 24 inches square. The calculation can be mental, but it belongs to the mathematics of quantity as long as what is mentally represented are actual physical quantities and actions on them. One is engaged in the mathematics of number only when the mental representation is of numbers as abstract entities, with properties defined relative to the system of numbers themselves, not to physical quantities. Earlier in this chapter I presented some examples of children beginning to reason in the mathematics of numbers, recognizing numbers as part of an abstract system rather than as names for physical quantities. However, as noted there, the capacity for such reasoning does not seem to come either early or easily; the passage from the mathematics of quantity to the mathematics of number appears to be a difficult one that may require more explicit attention in instruction than is now normally accorded to it.

Although the mathematics of numbers begins the passage to formal mathematics, it is not all that must be achieved. In the mathematics of numbers neither operations on numbers nor relations between numbers have yet become the objects of reasoning. Operations in the mathematics of numbers are like transitive verbs. They describe actions that can be performed *on* numbers. But they are not themselves objects with properties, objects on which actions can be taken. Similarly, in the mathematics of numbers one can describe relations *between* numbers, but the relations are essentially adjectives describing properties of the numbers. They are not themselves noun-like objects with properties that can be reasoned about. We thus need one more layer of mathematics to complete our story.

This final layer is the *mathematics of operators.* In the mathematics of operators operations behave like nouns. They can be reasoned *about*, not just applied. For example, it can be argued that the operation of *addition by combining* is always commutative, no matter what pair of numbers is composed. In the mathematics of operators, operations can also be operated *on.* The operation of *adding 4*, for example, can be composed with another operation of *subtracting 3* and this composition recognized as equivalent to an operation of *adding 1.* This equivalence it perfectly general; it holds no matter which particular numbers are operated on. Operations have now become, like numbers before them, mental entities, actual mathematical objects to reason about. A similar transformation of relations between numbers occurs at this layer. A *difference of 3*, for example, can be understood as a property of the pair, 11 and 8. Differences can be compared, so that one can recognize that [11 - 8] is less than [24 - 20], or even operated on so that [11 - 8] can be subtracted from [24 - 20]. Similarly, a multiplicative relation, *x 4*, can be recognized as equivalent whether one is describing the pairs [2,8], [15,60] or [11,44]. This kind of reasoning about relations as mental objects is what it takes to understand functions.

According to the layers theory, these four kinds of mathematics--of protoquantities, of quantities, of numbers, and of operators--are genetically related to one another. Reasoning about quantities--with numbers functioning as adjectives and operations as verbs--is built on a foundation of schemas for reasoning protoquantitatively about the relations between amounts of physical material. Once children become fluent with the mathematics of quantities, that is once they can reason easily about actions on numerically quantified physical material, they can begin to lift the numbers out of the physical quantity relations and treat them as objects in their own right. In this way the mathematics of numbers is developed. Gradually, over time and with extensive practice, numbers become nouns and the mathematics of numbers begins--with operators still functioning as verbs, but now describing actions on the mental entities of specific numbers. Only after extensive practice in operating on numbers as objects will the mathematics of operators, in which operations and relations themselves become objects, be constructed.

None of this should be taken to suggest an all or none, stage-like form of development. There is no reason to suppose that people pass all at once from a protoquantitative way of thinking to the mathematics of quantities or from the mathematics of quantities to the mathematics of numbers. There may be some developmental limits--of the kind developed by Case (1985) and Fischer (1980), for example--on how many chunks of information can be thought about at once, and this will be a brake of sorts on the pace at which new objects and properties can be incorporated into the mental system. But most evidence suggests that mental objects are built up specific bit by specific bit, rather than emerging in discrete stages. So, for example, children may be doing the mathematics of quantity on dimensions such as manyness and length, while still reasoning only protoquantitatively about weight or speed. Further, they may convert small integers (e.g., 1, 2, 3, and 4) into mental entities and perform the mathematics of number on them, while still using higher numbers only as descriptors of amounts of physical material. Similarly, operators may be transformed from verbs to nouns one at a time. Children can reason about the commutativity of addition, over all numbers, well before they can reason about multiplicative functions. Furthermore, as each new kind of number--positive integers, negative integers, fractions, etc.--is encountered, it is likely that learning will entail passage through the successive layers of the mathematics of protoquantities, quantities, numbers and operators. Thus, at any given moment a child can be functioning at several different layers of mathematical thought.

There is another important way in which the layers of mathematical knowledge described here differ from the usual notion of developmental stages. In passing to a higher layer of mathematical reasoning, the earlier layers are not discarded, but remain part of the individual's total knowledge system. People operating at a higher layer can use their knowledge of lower-layer referential meanings to relate their abstract mathematical knowledge to practical situations. Or they may reason in a partly formal (mathematics of operators) manner and partly physical (mathematics of quantities) manner on different parts of the same complex problem. Engineers sometimes reason protoquantitatively about physical systems (e.g., Forbus, 1988; deKleer & Brown, 1985), using the conclusions reached about how quantities should change or relate to one another to constrain and check the results of more formal calculations.

These relationships among the layers of mathematical reasoning have important implications for instruction. They imply that attempts to skip over or press a child to rush through layers of mathematical thinking are likely to limit eventual mathematical competence. Earlier layers are part of the foundation of higher ones; performances that appear to be at high levels of abstraction but have not been built on a foundation of more basic layers will not be stable and robust. If any of the layers is skipped--for example by trying to drill children on number facts without developing number relationships in the context of the mathematics of quantity--the result may be behavior that looks superficially like more advanced mathematical behavior, but which can be destroyed by demands for reasoning beyond what has been explicitly practiced. It is that kind of behavior that we are probably seeing in the buggy rules for arithmetic and algebra discussed earlier. Furthermore, individuals who have practiced manipulating numbers without earlier reference to quantities are likely to have difficulty in using their number knowledge to solve problems about the physical world.

The Psychological Origin of Mathematical Principles

The theory of layers of mathematical knowledge provides new lenses for examining the evidence, presented earlier, of children's intuitive understanding of arithmetic. It is especially instructive to review the mathematical principles that young children seem to appreciate to determine how they might be rooted in successive layers of mathematical thinking and where difficulty might be encountered in passing to the higher layers of mathematical thinking.

Principles based on a part/whole schema. Consider first a set of principles that can be viewed as elaborations of a protoquantitative part/whole schema. Children know from their experience of the physical world how material comes apart and reassembles. Before they can reliably quantify physical material, that is when they are still functioning at a mathematics of protoquantities, they know that a whole quantity (W) can be cut up into two or more parts (P_1, P_2, P_3,...), that the parts can be recombined to make the whole, that the order in which the parts are combined does not matter in reconstituting the original amount. This knowledge can be represented as a set of protoquantitative equations:

(1) $P_1 + P_2 + P_3 = W$

(2) $P_1 + P_2 = P_2 + P_1 = W$

(3) $[P1 + P2] + P3 = P1 + [P2 + P3] = W$

Equation (2) is a protoquantitative version of the commutativity of addition principles, equation (3) of the associativity of addition principle.

As children apply their counting skills in situations that earlier were reasoned about only protoquantitatively, they begin to develop the mathematics of quantities. In the process, the part/whole schema becomes quantified. All of the relationships between whole and parts that were present in the protoquantitative schema are maintained, but now the relations apply to specific, quantified amounts of material. As a result, children can now reason using quantified equations such as:

(4) 3 apples + 5 apples + 4 apples = 12 apples

(5) 4 cakes + 7 cakes = 7 cakes + 4 cakes

(6) [3 apples + 5 apples] + 4 apples = 3 apples + [5 apples + 4 apples]

Equations (5) and (6) constitute versions of the commutativity and associativity principles within the mathematics of quantities.

As numbers are lifted out of their external referential context and the mathematics of numbers begins, the part/whole schema can organize knowledge about relations among numbers:

(7) $3 + 5 + 4 = 12$

(8) $4 + 7 = 7 + 4$

(9) $[3 + 5] + 4 = 3 + [5 + 4]$

Equations (8) and (9) constitute versions of commutativity and associativity at the mathematics of numbers layer.

Finally, at the layer of mathematics of operators, attention switches from actions on particular numbers to more general relations between numbers. Commutativity and associativity are *always* true for addition, no matter what the numbers. Thus:

(10) $n + m = m + n$

(11) $[n + m] + p = n + [m + p]$

The important thing to note about this sequence of equations is that the very same *relations* remain in place through all four layers, but the *objects* change at successive layers: first unquantified amounts, then quantified amounts, then specific numbers, and finally numbers in general. The same rooting of principles in protoquantitative knowledge and reasoning can be seen for more complex principles as well. The principle of complementarity of addition and subtraction (additive inverse), for example, also begins with the protoquantitative part/whole schema, as the following set of equations shows:

(12) $P1 + P2 = W$

(13) $W - P1 = P2$

(14) $W - P2 = P1$

These equations express the basic logic of part/whole relations, namely that if a whole is split into two parts, and one part is removed, the other part is what remains. The same logic is maintained under the mathematics of quantities:

(15) 4 cakes + 7 cakes = 11 cakes

(16) 11 cakes - 4 cakes = 7 cakes

(17) 11 cakes - 7 cakes = 4 cakes

and of numbers:

(18) $4 + 7 = 11$

(19) $11 - 4 = 7$

(20) $11 - 7 = 4$

The logic becomes fully general, applying to any numbers, under the mathematics of operators:

(21) $m + n = p$

(22) $p - m = n$

(23) $p - n = m$

Principles based on an increase/decrease schema. The protoquantitative increase/decrease schema organizes children's knowledge of the effects of growth or shrinkage in a quantity or the results of adding or taking away material from an established amount of that material. On the basis of this schema, children are able to conclude that when something is added to a starting amount the new amount is greater than before, when something is taken away from a starting amount the new amount is less than before, and that when nothing is added or taken away there is the same amount as at the start.

At the mathematics of quantities layer a quantified increase/decrease schema plays an important role in early story problem solutions. One of the kinds of problems that has been shown to be easy for even kindergartners to solve is one-step stories about *changes* in sets. Few primary children have much difficulty even with two-step stories such as:

> "John had 17 marbles. He lost 11 of them in a first game and 4 in a second game. How many does he have left?"

Children solve problems of this kind by physically or mentally representing a set of 17 objects, removing 11 of them, removing another 5, and counting the remainder.

In this problem the numbers 11 and 4 specify the size of decreases, actions on established sets. This can be symbolized as:

(24) 17 marbles (-11 marbles) (-4 marbles) = 2 marbles,

where the parentheses indicates that the numbers quantify the operations of taking away. In the mathematics of numbers the equation would become:

(25) 17 (-11) (-4) = 2

But only in the mathematics of operators would it be possible to compose the two transformations as in:

(26) (-11) + (-4) = (-15)

Children have a great deal of difficulty for some time in thinking of the transformations as objects that can themselves be combined as in equation (26). For many years they think of numbers as describing the cardinalities of sets, rather than as quantifications of transformations increases or decreases). Evidence of this preference comes from a study by Resnick, Cauzinille-Marmeche and Mathieu (1987), who asked 11 through 14-year-olds to make up stories that would yield expressions such as 17-11-4. Many of the younger students created stories such as:

The boy has 17 marbles. His friend would have 11, and another would have 4, just a little batch...

...there are 17 marbles in the morning, the marbles at noontime, and 4 in the evening...

Older children usually could successfully create stories that would produce the expression, 17-11-4, but they often could not combine the two transformations into a total "loss" of 15 marbles except by doing successive subtractions on the starting set. Thus, they would perform a sequence of operations such as 17 - 11 = 6; 6 - 4 = 2; 17 - 2 = 15 to find an answer. But they could not solve the problem directly by composing the two transformations, as in equation 26. Still less could they reason generally about composition of operators as in:

(27) $(-m) + (-n) = -(m + n)$.

The protoquantitative increase/decrease schema joins with protoquantitative part/whole to provide the early intuitive foundation for another important principle, compensation and equivalence. The combination of the two schemas allows children to reason about the effects of changes in parts and wholes on one another. Figure 6 shows the kinds of inferences that are possible under a wholly protoquantitative form of reasoning. As the figure shows, children are able to conclude that if something is added to one of two parts while the other stays constant the whole must increase. Or, if the whole is decreased, at least one of the parts (perhaps both) must also decrease. The most complex reasoning enabled by the combination of the protoquantitative part/whole and increase/decrease schemas concerns how the whole can be maintained unchanged by compensating changes in the two parts:

(28) $(P1 + A) + (P2 - A) = W$

That is, if an amount is added to part 1 and the same amount is removed from part 2, then the whole remains unchanged.

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insert Figure 6 here

In the mathematics of quantities, the compensation schema takes on specific values. For example, equation (15) can be modified by addition and subtraction of the same amount of cakes from the two parts without changing the whole:

(29) [4 cakes + 2 cakes] + [7 cakes - 2 cakes] = 11 cakes

Eventually, when the mathematics of operators is reached, the principle of compensation becomes general, as in,

(30) $[m + a] + [n - a] = [m + n] + [a - a] = m + n$.

General relationships of this kind are reflected in the rules for derived fact arithmetic described by Putnam, deBettencourt, and Leinhardt (1990) and shown in Table 1.

Principles based on a compare schema. The protoquantitative compare schema is, very probably, children's earliest form of mathematical knowledge. It permits even infants to make comparative judgments about amounts of physical material (see Resnick, 1983). The compare schema is called upon in both part/whole and increase/decrease protoquantitative reasoning, permitting judgments about the relative magnitude of parts and wholes and about amounts of material before and after transformations. Almost as soon as they learn to count, children are able to judge the relative magnitudes of numbers, even without referential quantities of material. They can decide, for example, that 7 is more than 4, or 3 is less than 5. But it is some years before most children are able to quantify differences, to say how *much* more 7 is than 4.

Solving comparison story problems, such as, "There are 5 worms and 8 birds. How many more birds than worms?" is difficult for most children until the age of 8 or 9 (Riley & Greeno, 1988). This kind of problem requires establishing a *quantified difference relationship* between the two *numbers,* 5 and 8. This can be done only at the mathematics of numbers layer, for a difference can be quantified only between like objects, and this means forgetting about the worms and birds and thinking only about the 5 and 8. A further development of thinking about differences comes with the mathematics of operators. At this layer of mathematical thinking, where differences are mental entities that can be compared to one another, it becomes possible to construct *equivalence classes of differences.* This permits recognition that many different pairs of numbers have the same difference, as in,

(31) $[2 - 0] = [3 - 1] = [4 - 2] = [20 - 18] = [133 - 131] = \ldots.$

More generally, this leads to a principle of *conservation of differences* by addition or subtraction of the same amount to the minuend and the subtrahend of a subtraction pair:

(32) $n - m = [n + a] - [m + a]$

(33) $n - m = [n - a] - [m - a]$

This principle underlies certain subtraction procedures. For example, in many countries children are taught to subtract in the following way:

$$\begin{array}{lcl} 132 & & 13^{1}2 \text{ (add 10 to minuend)} \\ & \text{is changed to:} & \\ \underline{-28} & & \underline{-38} \text{ (add to subtrahend)} \end{array}$$

Mental arithmetic methods sometimes apply this principle, as well. For example, to take advantage of a well-known subtraction fact, an individual might convert [9 - 4] to [10 - 5]. However, children are far less likely to make this kind of conversion than they are conversions based on additive compensation, as in equation (30) (Putnam, deBettencourt, & Leinhardt, 1990). Thus it appears that additive pairs become mental objects earlier than differences.

Teaching Mathematics on the Basis of Intuitive Knowledge

The theory of layers of mathematical knowledge helps to provide a more specific meaning for the idea of teaching mathematics on the basis of children's intuitive knowledge. Broadly stated, to teach on the basis of intuitive knowledge requires identifying, for each broad class of mathematical concepts, which layer children are already functioning in. This constitutes their intuitive knowledge at that moment of development. One then arranges situations of practice and discussion that will help children elaborate their schemas at successively higher layers. On the basis of research results now available, it is possible to make quite specific suggestions about how to apply this broad strategy to the earliest years of school mathematics teaching.

Elaborating and Developing the Schemas

Quantifying the protoquantitative schemas. Most five and six year olds come to school with well developed protoquantitative knowledge, at least for the additive composition aspects of physical material. That is, they already know and can apply the schemas such as those expressed in equation (1) through (3) and (12) through (14) and (28). An important initial task for school mathematics is to help children quantify their protoquantitative schemas, and thus enter the layer of mathematics of quantities. A primary means for doing this is to provoke and support counting-based solutions to problems that involve the protoquantitative schemas. Most children starting school know how to quantify at least small sets of objects (say, up to 10) by counting them. They will count whenever directly asked the question "How many?" But for some time after they learn how to count sets children do not spontaneously count to solve problems involving quantity relations such as comparing, combining or changing. By engaging children in counting activities *in the context of problems they already know how to solve protoquantitatively*, teachers can help children invest counting with all of the meaningful relations inherent in the protoquantitative schemas.

It is unlikely that drill on number facts will be useful in speeding up the process of quantifying the protoquantitative schemas. Such drill removes numbers from the referential context that evokes the schemas and is likely to promote fragile knowledge that children are unable to apply in problem solving. Rather than memorization drills, what is needed is extensive practice in solving well understood quantity problems. At first it will be helpful to focus on problems in which the material to be quantified is actually present. Just a little later stories about very familiar types of situations can be successfully used. These stories can be presented by the teacher, or brought to school or made up by the children. Children can also be encouraged to find quantification problems in their everyday lives. For children to effectively use counting in these situations, their counting skill must be sufficiently developed that they can count objects more or less *automatically*, without much conscious attention. Otherwise attention to the goal of counting will drive from memory the protoquantitative relations. This may mean that some children at school entrance will be able to solve by themselves only problems involving very small quantities. However, even these children can benefit by participating "peripherally" (cf. Lave, in press) in classroom discussions in which others are solving larger quantity problems. Such participation can alert them to the idea that quantitative solutions are possible for large quantities, indeed for any quantities. This is an important form of *scaffolding* that provides children with frameworks for using counting skills even before the counting skills are fully developed.

All of the various classes of story problems described in the now extensive literature on addition and subtraction stories need to be included in these early steps toward quantification of the schemas. Research establishing a sequence of difficulty for these problems (e.g., Riley & Greeno, 1988) can be used to guide the sequence of introduction. But there has not been enough research on the deliberate introduction of these classes of problems to strictly constrain an instructional sequence. Indeed, the notion of learning through scaffolded situational participation would argue against very much deliberate sequencing in the introduction of problems. In any group of children there is likely to be wide variation in ability to solve quantified problems. But if problems are chosen for which most children already have the necessary protoquantitative schemas, all children will be able to solve the problem at some layer of mathematical thinking. Some will be able to do it on their own only protoquantitatively--that is, in terms of *more* and *less*, but not *how many*. Others will be able to solve it by directly counting the objects or representations of objects in the story (such as fingers, tick marks, or other manipulatives). Still others will be able to solve the problems mentally using number facts and relations between numbers that have by now become intuitive for them.

In a class discussion of the problem and various solution strategies, the least developed children will observe others solve the problem quantitatively--through counting or even number relationships--and can be coached through counting solutions themselves. For this to work, some method of making different children's solution methods visible to one another is needed. In one successful program (Resnick, Bill, Lesgold, & Leer, in press) this is accomplished by having children solve the problem privately, then share solutions in small groups, then describe and justify their solutions to the whole class. In this way, a number of different solutions, each treated as a valid option for the situation are collected. Children can also be asked to record their own and others' solutions. This further heightens children's awareness of the variety of solutions possible.

Strengthening incipient schemas. Another important goal of the first few years of schooling is the further development and strengthening of protoquantitative and very basic quantified schemas. This includes both developing previously missing schemas and developing heightened awareness--through language--of schemas the children already possess. When we say that children "have" the combine or the increase/decrease protoquantitative schema, we mean that they can apply it to reach judgments about relations among amounts of material. But some children may be good at such applications without being very practical at *talking about* protoquantitative relations. That is, they can make judgments, but not justify them, even in ordinary language. This means that while they can use their schemas, they cannot reason *about* them (e.g., by saying "a whole is always bigger than its parts" or "You always have less if someone takes some away"). Absent this *meta* ability with respect to their protoquantitative knowledge children are missing a major element for the eventual objectification of their knowledge about quantities and operations on them. Children can learn these self-reflective, meta abilities the way they learn anything else--by participation in situations where these activities are a normal way of behaving. Accordingly, the primary school classroom ought to establish itself as a situation in which talk--first about protoquantitative relations, later about quantified relationships--is a normal way of behaving.

In addition to talk about the additive protoquantitative schemas the children bring with them to school, the primary grade classroom can be a place for development of protoquantitative and quantified schemas that will be the basis for eventual elaboration of multiplication, division, fraction, ratio and proportion concepts (cf. Resnick & Singer, in press). Schemas of positive and inverse covariation, for example, can be developed through problems involving ordering of physical material. Graphs and charts showing the quantities rising and falling together would be within the capabilities of relatively young children to create and discuss. It is not necessary that children notice or discuss the numerical function that relates the two quantities at this early stage of reasoning. Children's protoquantitative understanding of repetition can also be developed at an early stage. This is a precursor of multiplication (cf. Nesher, this volume). Repetition of a quantity is at the heart of measurement (the blue stick can be moved along the desk top 10 times). Thus all kinds of measurement activities (number of lengths, number of cups of liquid, etc.) serve to develop a repetition schema. Finally there is the inverse of repetition: partitioning into equal units, which is an early form of division. Here, problems of sharing are natural for children: maintenance of equivalent shares is for them a natural way to make things fair.

From arithmetic on quantities to arithmetic on numbers. Once the basic additive schemas are quantified, the next step is for the numbers to become detachable from material and function as mental entities in their own right--that is, for the numbers to change status from adjectives to nouns. The objectification of numbers is not an all-or-none affair. Rather, it happens over a long period of time, several years at least. Furthermore, reaching the stage of arithmetic on numbers does not mean that arithmetic on quantities will be left behind, any more than ability to quantify the basic additive schemas means that children never reason protoquantitatively. Instead, each level of reasoning is called upon in situations appropriate to it. The ability to move back and forth easily and appropriately between protoquantitative, quantitative and more abstract number reasoning is a hallmark of the mathematically competent.

Children entering school already know something about numbers as abstract entities. They understand that when counting, the words in the number string ("one, two, three . . .") do not *refer* to the objects one points to as one says them--since the objects can be pointed to in any order with the same outcome for how many objects in the total set (Gelman & Gallistel, 1978; Greeno, Riley, & Gelman, 1984). Children also know something about the *properties* of numbers even before school entry. In particular, they know that numbers have properties of order and magnitude. As young as 4 years, children can promptly answer questions about relative magnitude of numbers, such as "which is larger, 5 or 8?" What is more, they do so in a way that shows that they are not just counting through the number string ("5, 6, 7, 8–so 8 is larger") to derive their answers; but instead can directly access the information about the magnitudes of 5 and 8. They behave as if they had a "mental number line" (Resnick, 1983).

Yet preschool children's mental numbers are impoverished entities compared to what they will later become. Their mental numbers have the properties only of order and magnitude. Furthermore, the magnitude property is only protoquantitative. Preschool children cannot say how *much* larger one number is than another, except using terms such as *a lot bigger* or *a little bigger*. A second major task of the first two or three years of school, accordingly, is to help children acquire a much richer knowledge of the properties of numbers as mental entities.

The first property of number likely to become accessible to children on the basis of their quantified part/whole schema is the property of numbers as *compositions* of other numbers. Knowledge of this property will take substantial time to develop, because at the mathematics of numbers layer, the composition property applies not to numbers in general, but to each individual number. Furthermore, each number has many compositions in which it participates. All of these characteristics of individual numbers must be learned. As with any other learning, the route to mastery is extensive practice in situations that lend meaning to the activity engaged in. The kind of situations likely to yield this meaning for numbers are those that involve children in using the numbers *in the context of the quantified part/whole schema* and in *talking about* the compositional properties of the numbers used. To move toward the mathematics of numbers, the act of actually counting material must gradually disappear, so that attention can shift to properties of numbers as objects. But to keep the focus on composition it is still important that numbers be manipulated in the context of situations that evoke the compositional properties of material. This is why story problems--preferably ones with increasingly complex or multiple compositions and decompositions--should remain central to the curriculum. The focus on manipulative materials (actually countable objects) should be phased out. This will encourage children to find solutions based on their growing knowledge of how the numbers themselves compose and decompose. Children can be permitted to use manipulatives (or fingers) whenever they need to, thus only gradually, and at a rate individual children can set for themselves, eliminating counting.

An Elementary Classroom Program

Many of the examples of teaching used in the course of this chapter are drawn from the work of a teacher who has collaborated with our research group in developing a primary grades mathematics program based on the theory developed in this chapter. I describe the program here in somewhat schematized form as the instantiation of a set of six principles that have guided our thinking as the program has developed.

1. Develop children's trust in their own knowledge. Traditional instruction, by focusing on specific procedures and on special mathematical notations and vocabulary, tends to teach children that what they already know is not legitimately mathematics. To develop children's trust in their own knowledge *qua* mathematics, our program stresses the possibility of multiple procedures for solving any problem, invites children's invention of these multiple procedures, and asks that children explain and justify their procedures using everyday language. In addition, the use of manipulatives and finger counting insures that children have a .ay of establishing for themselves the truth or falsity of their proposed solutions. Figure 7 provides examples of multiple solutions given by second grade children to solve the same addition problem, *158 + 74*. These solutions in the figure illustrate the ways in which written notation and mental arithmetic are combined in the children's procedures.

Insert Figure 7 here

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2. Draw children's informal knowledge, developed outside school, into the classroom. An important early goal of the program is to stimulate the use of counting in the context of the compare, increase/decrease, and part/whole schemas in order to promote children's construction of the quantified versions of those schemas. This is done through extensive problem-solving practice, using both story problems and acted-out situations. Counting (including counting on one's fingers) is actively encouraged. Figure 2, discussed earlier in this chapter, gives an example of a typical class problem, showing how it can generate several solutions; the notations shown are copied from the notebook in which a child recorded the solutions proposed by several teams who had worked on the problem.

3. Use formal notations (identity sentences and equations) as a public record of discussions and conclusions. In order to move children toward the mathematics of numbers and operators, children's intuitive knowledge must be linked to the formal language of mathematics. By using a standard mathematical notation to record conversations carried out in ordinary language and rooted in well understood problem situations, the formalisms take on a meaning directly linked to children's mathematical intuitions. Figure 8 shows part of a typical teacher-led sequence in which children propose a solution to a story problem. The teacher carefully linked elements of the proposed solution to the actual physical material involved in the story (a tray of cupcakes) and an overhead schematic of the material. Only after the referential meaning of each number had been carefully established was the number written into the equation.

Figure 8 about here

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4. Introduce key mathematical structures as quickly as possible. As discussed earlier in this chapter, children's protoquantitative schemas already allow them to think quite powerfully about how amounts of material compare, increase and decrease, come apart and go together. In other words, they arrive in school with protoquantitative precursors of principles such as commutativity, associativity, and additive inverse already known. A major task of the first few years of school mathematics is to "mathematize" this knowledge--that is, quantify it and link it to formal expressions and operations. We thought that this could best be done by laying out the additive structures (e.g., for first grade: addition and subtraction problem situations, the composition of large numbers, regrouping as a special application of the part/whole schemas) as quickly as possible and then allowing full mastery (speed, flexibility of procedures, articulate explanations) of elements of the system to develop over an extended period of time. Guided by this principle, we found it possible to introduce addition and subtraction with regrouping in February of first grade. However, no specific procedures were taught; rather children were encouraged to invent and explain ways of solving multidigit addition and subtraction problems, using appropriate manipulatives and/or expanded notation formats that they developed.

A program built around this principle constitutes a major challenge to an idea that has been widely accepted in educational research and practice. This is the notion of learning hierarchies, specifically that it is necessary for learners to master simpler components before they try to learn complex skills. According to theories of hierarchical and mastery learning, children should, for example, thoroughly master single digit addition and subtraction before attempting multidigit procedures, and they should be able to perform multidigit arithmetic without regrouping smoothly before they tackle the complexities of regrouping. We developed instead a *distributed* curriculum in which multiple topics are developed all year long, with increasing levels of sophistication and demand, rather than a strictly sequential curriculum. To convey the flavor of the process, Figure 9 shows the range of topics planned for a single month of the second grade program. All topics shown are treated at changing levels of sophistication and demand throughout the school year. This distributed curriculum discourages decontextualized teaching of components of arithmetic skill. It encourages children to draw on their existing knowledge framework (the protoquantitative schemas) to interpret advanced material, while gradually building computational fluency.

Insert Figure 9 here

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5. Encourage everyday problem finding. Every day problem finding means doing arithmetic every day, not only in school but also at home and in other informal settings. This is important for two reasons. First, children need massive practice in applying arithmetic ideas, far more than the classroom itself can provide. For this reason we thought it important to encourage children to find problems for themselves that would keep them practicing number facts and mathematical reasoning. Second, it is important that children come to view mathematics as something that can be found everywhere, not just in school, not just in formal notations, not just in problems posed by a teacher. We wanted to get children in the habit of noticing quantitative and other pattern relationships wherever they were and of posing questions for themselves about those relationships. Two aspects of the program represent efforts to instantiate this principle. First, the problems posed in class are drawn from things children know about and are actually involved in. Second, homework projects are designed so that they use the events and objects of children's home lives: for example, finding as many sets of four things as possible in the home; counting fingers and toes of family members; recording numbers and types of things removed from a grocery bag after a shopping trip. From child and parent reports, there is good, although informal, evidence that this strategy works. Children in the program are noticing numbers and relationships and setting problems for themselves in the course of their everyday activities.

6. Talk about mathematics, don't just do arithmetic. Talk about mathematical ideas is an essential means of helping children construct the conceptual entities that will allow them to reason about numbers and operators. To encourage this talk, our program uses a combination of whole-class, teacher-led discussion and structured small group activity by the children. In a typical daily lesson, a single relatively complex problem is presented on the chalkboard. The first phase is a class discussion of what the problem means--what kind of information is given, what is to be discovered, what possible methods of solution there are, and the like. In the second phase, teams of children work together on solving the problem, using drawings, manipulatives, and role playing to support their discussions and solutions. The teams are responsible not only for developing a solution to the problem, but also for being able to explain why their solution is a mathematically and practically appropriate one. Figure 10, a four-minute segment of a third grade team's conversation as they work independently on a problem, shows how linguistic interpretation and development of manipulative displays interact in the children's work. In the third phase of the lesson, teams successively present their solutions and justifications to the whole class, and the teacher records these on the chalkboard. The teacher presses for explanations and challenges those that are incomplete or incorrect; other children join in the challenges or attempt to help by expanding the presented argument. By the end of the class period, multiple solutions to the problem, along with their justifications, have been considered, and there is frequently discussion of why several different solutions could all work, or why certain ones are better than others. In all of these discussions, children are permitted to express themselves in ordinary language. Mathematical language and precision are deliberately not demanded in the oral discussion. However, the equation representations that the teacher and children write to summarize oral arguments provide a mathematically precise public record, thus linking everyday language to mathematical language.

Insert Figure 10 here

The program just described, now in its third year of operation has produced substantial gains in children's performance on standardized math tests, as well as evidence of comprehension of basic mathematical principles and ability to solve relatively complex problems. But the program is not just an apparently successful way of teaching early mathematics. It embodies some fundamental challenges to dominant assumptions about learning and schooling. As we worked to develop this program, we realized that a new theoretical direction was increasingly dominating our thinking about the nature of development, learning, and schooling. This is the view, shared by a growing minority of thinkers in the various disciplines that comprise cognitive science, that human mental functioning must be understood as fundamentally situation-specific and context-dependent, rather than as a collection of context-free abilities and knowledge. This apparently simple shift in perspective in fact entails reconsideration of a number of long-held assumptions in both psychology and education.

Until recently, educators and scholars have defined the educational task as one of teaching specific knowledge and skills. As concern shifted from routine to higher order or thinking abilities, cognitive scientists developed more complex definitions of the skills to be acquired and even introduced various concepts of *meta* skill in the search for teachable general abilities. But most of us had continued to think of our major concern as one of identifying and analyzing particular skills of reasoning and thinking and then finding ways to teach them, on the assumption that successful students would then be able to apply these skills in a wide range of situations.

As we developed the math program described here, we found ourselves less and less asking what constitutes mathematics *competence* or *ability* for young schoolchildren, and more and more analyzing the *situations for performance* afforded by the mathematics classroom. This is why we focused so heavily on choosing problems on the basis of the mathematical principles they might illustrate and on developing forms of classroom conversation designed to evoke public reasoning about these principles. We began to define our task as creating the kind of *cognitive apprenticeship* called for by Collins, Brown, and Newman (1989) in a recent influential paper. That meant thinking of mathematics as a form of cultural practice in which children could participate, rather than as a bounded, static body of knowledge. This view does not deny that children engaging in mathematical practice must be knowledgeable and skillful in many ways. However, our emerging perspective led us to focus far less on the design of "lessons" than on the development of sequences of problem solving situations in which children could successfully participate. By creating an apprenticeship environment for mathematical thinking in which children could participate daily, we expected children to acquire thereby not only the skills and knowledge that expert mathematical reasoners possess, but also a social identity as a person who is able to and expected to engage in such reasoning. What made this possible was the strategy of beginning with children's protoquantitative intuitions and gradually expanding to higher layers of mathematical reasoning.

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Figure 1. Pitt's written display of his method for adding 60 and 35.

Figure 2. Multiple solutions illustrating the complementarity of addition and subtraction.

Figure 3. David's protoquantitative transfer schema.

Figure 4. The invention of an algebra malrule.

Figure 5. The "Holding Hands" example: Successive numbers in each string are paired.

Countersuggestion to "Holding Hands" example: Numbers are matched by values.

Figure 6. Inferences from combined protoquantitative increase/decrease and part/whole schemas.

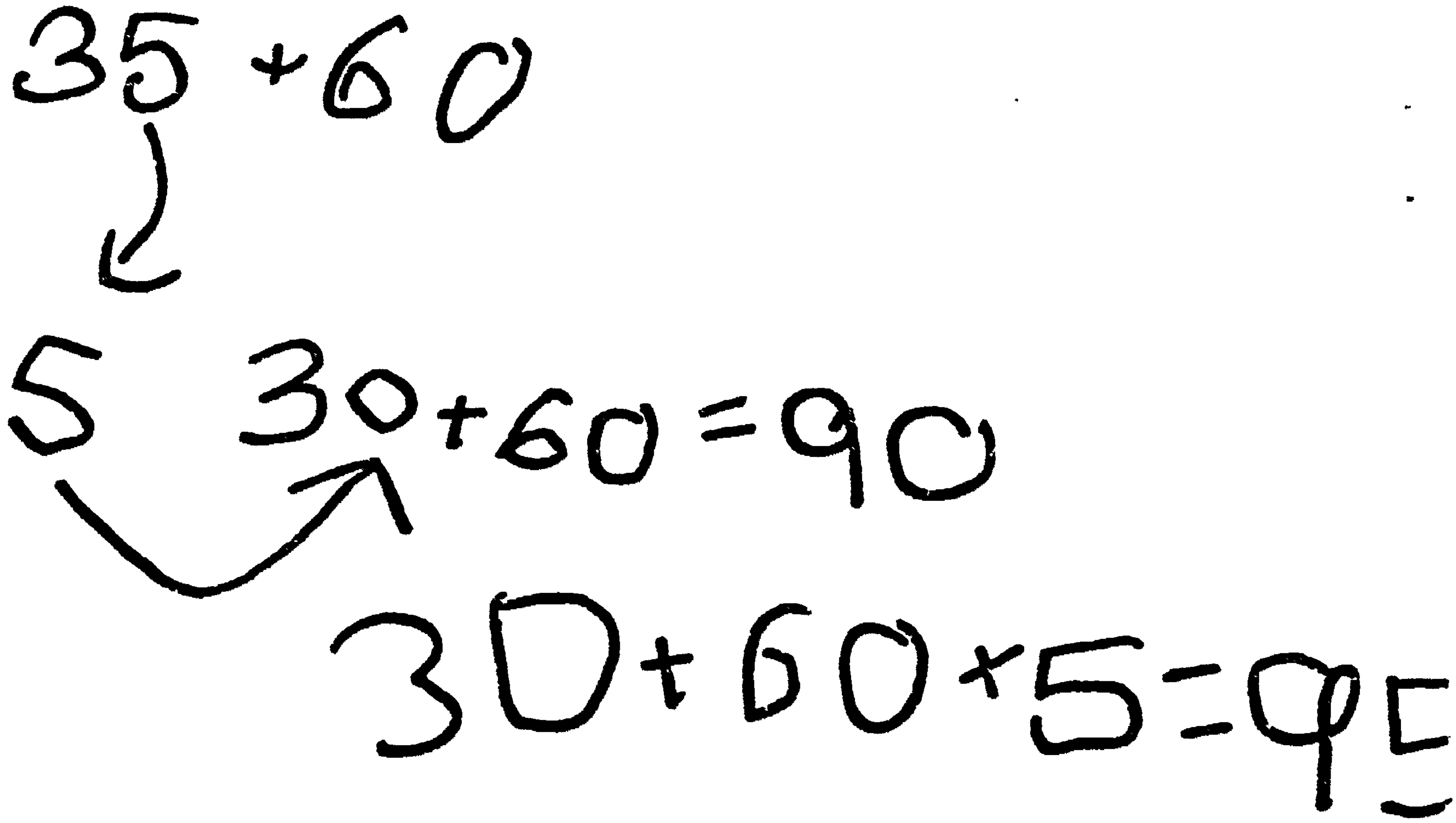
Figure 7. Examples of several second graders' solutions to a computational problem.

Figure 8. Part of a whole-class discussion of a story problem.

Figure 9. Topic coverage planned for a single month of grade 2.

Figure 10. Excerpt of a third grade team's conversation.

PITT'S WRITTEN DESCRIPTION


35 + 60
5
30 + 60 = 90
30 + 60 + 5 = 95

Mary told her friend Tonya that she would give her 95 barrettes. Mary had 4 bags of barrettes and each bag had 9 barrettes. Does Mary have enough barrettes?

The class first developed an estimated answer. Then they were asked, "How many more does she need?" The solutions below were generated by different class groups.

Group 1 first solved for the number of barrettes by repeated addition. Then they decomposed 4 x 9 into 2 x 9 plus 2 x 9. Then they set up a missing addend problem, 36 + 59, which they solved by a combination of estimation and correction.

Group 2 set up a subtraction equation and then developed a solution that used a negative partial result.

Group 4 began with total number of barrettes needed and subtracted out the successive bags of 9.

Est. $4 \times 10 = \boxed{40}$ NO

1-24-90

#1 $9+9+9+9=36 \longrightarrow$ $4 \times 9 = 36$

$2 \times 9 = 18$

$2 \times 9 = 18$

$18 + 18 = 36$

$\boxed{60}$

$36 + \underline{59} = 95$

$36 + \boxed{60} = 96$

$96 - 1 = 95$

$60 - 1 = 59$

#2 $95 - 36 = \underline{59}$

$90 - 30 = 60$

$5 - 6 = -1$

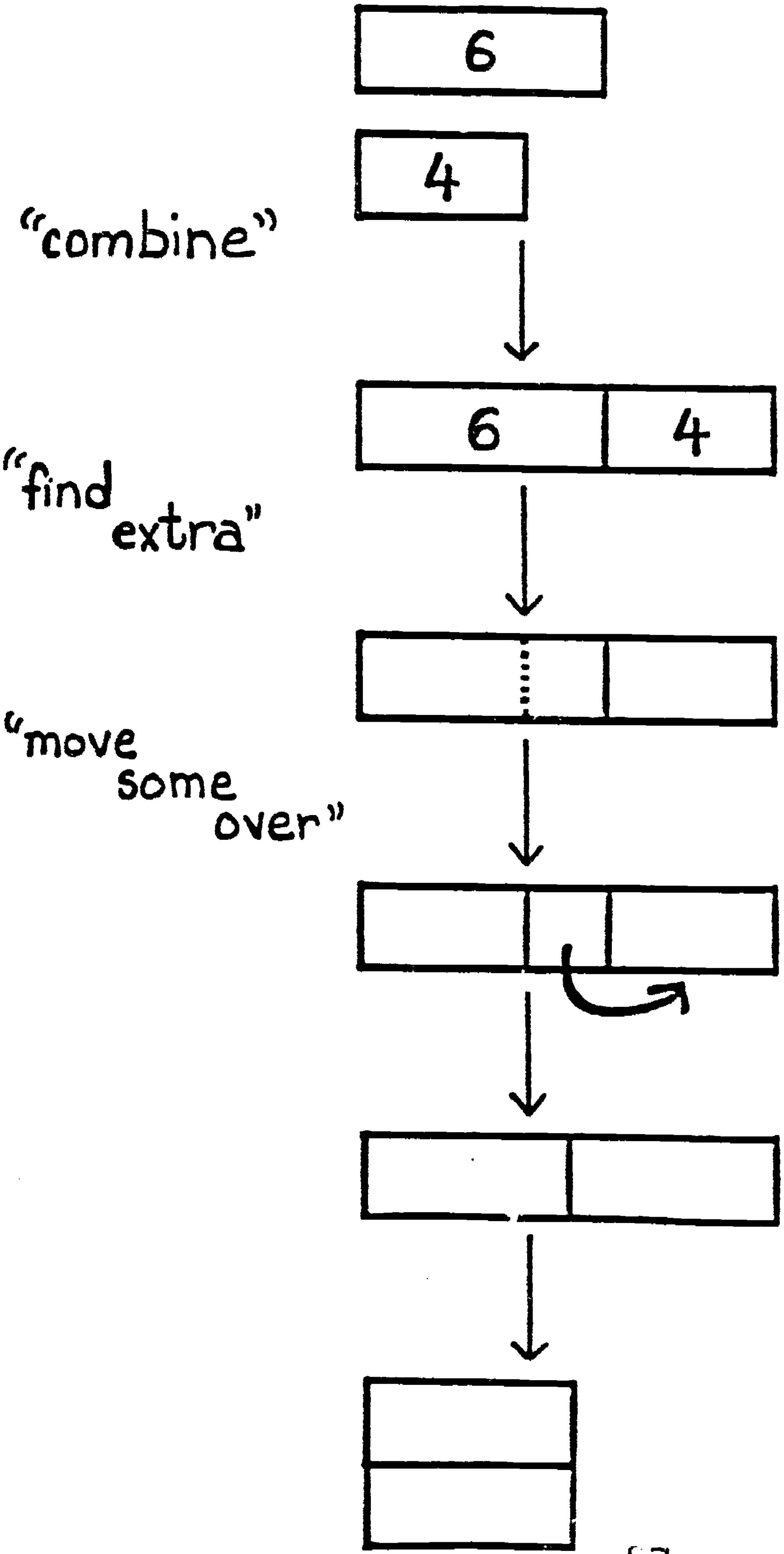
$60 - 1 = 59$

#4 $95 - 9 = 86$

$86 - 9 = 77$

$77 - 9 = 68$

$68 - 9 = \boxed{59}$


6
4
"combine"
6
4
"find extra"
"move some over"

1. The correct rule as taught:

$$a \times (b + c) = (a \times b) + (a \times c)$$

2. Prototype created by generalizing over operator signs:

$$a \square (b \triangle c) = (a \square b) \triangle (a \square c)$$

3. Incorrect rules created from the prototype.

$$a + (b \times c) = (a + b) \times (a + c)$$

$$\sqrt{b + c} = \sqrt{b} + \sqrt{c}$$

$$\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5\ldots \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 2 & 4 & 6 & 8 & 10\ldots \end{array}$$

$$\begin{array}{cccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10\ldots \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ & 2 & & 4 & & 6 & & 8 & & 10\ldots \end{array}$$

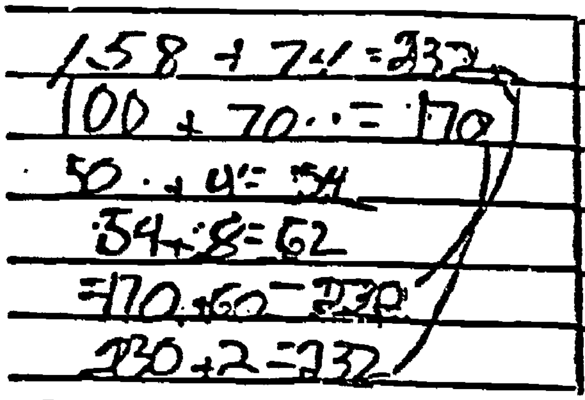
A. Changes in the Whole

Change in One Part	Change in the Other Part		
	+	0	−
+	+	+	?
0	+	0	−
−	?	−	−

B. Changes in One Part

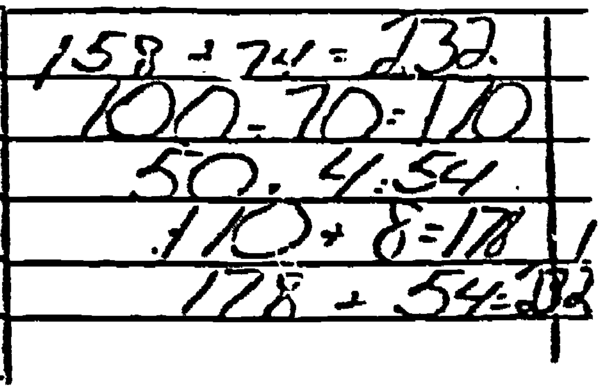
Change in the Whole	Change in the Other Part		
	+	0	−
+	?	+	+
0	−	0	+
−	−	−	?

A



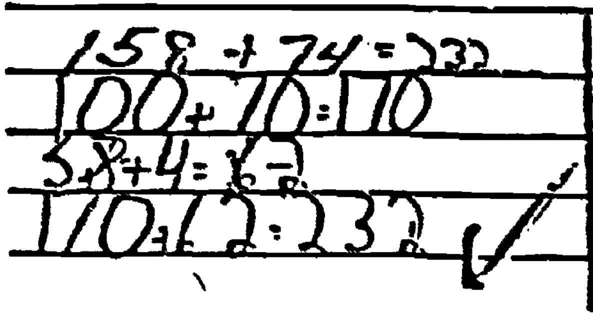
158 + 74 = 232
100 + 70 = 170
50 + 4 = 54
54 + 8 = 62
170 + 60 = 230
230 + 2 = 232

B



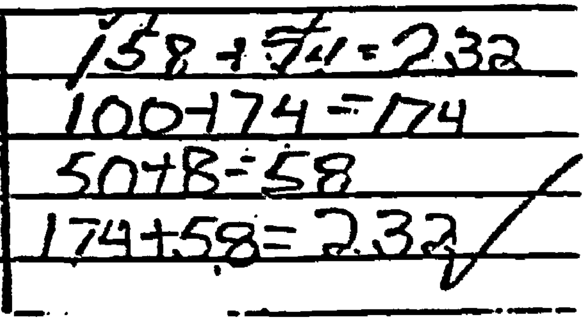
158 + 74 = 232
100 - 70 = 170
50 + 4 = 54
170 + 8 = 178
178 + 54 = 232

C



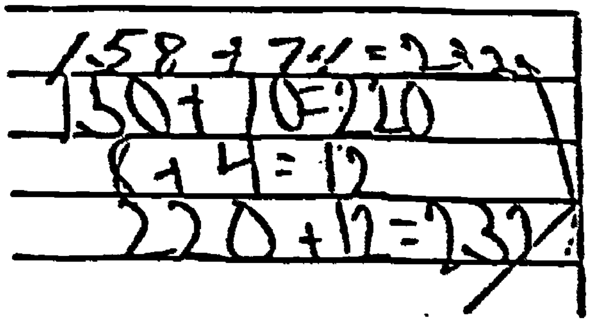
158 + 74 = 232
100 + 70 = 170
58 + 4 = 62
170 + 62 = 232

D



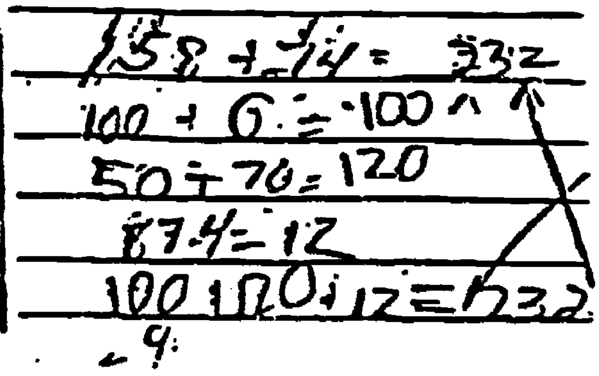
158 + 74 = 232
100 + 74 = 174
50 + 8 = 58
174 + 58 = 232

E

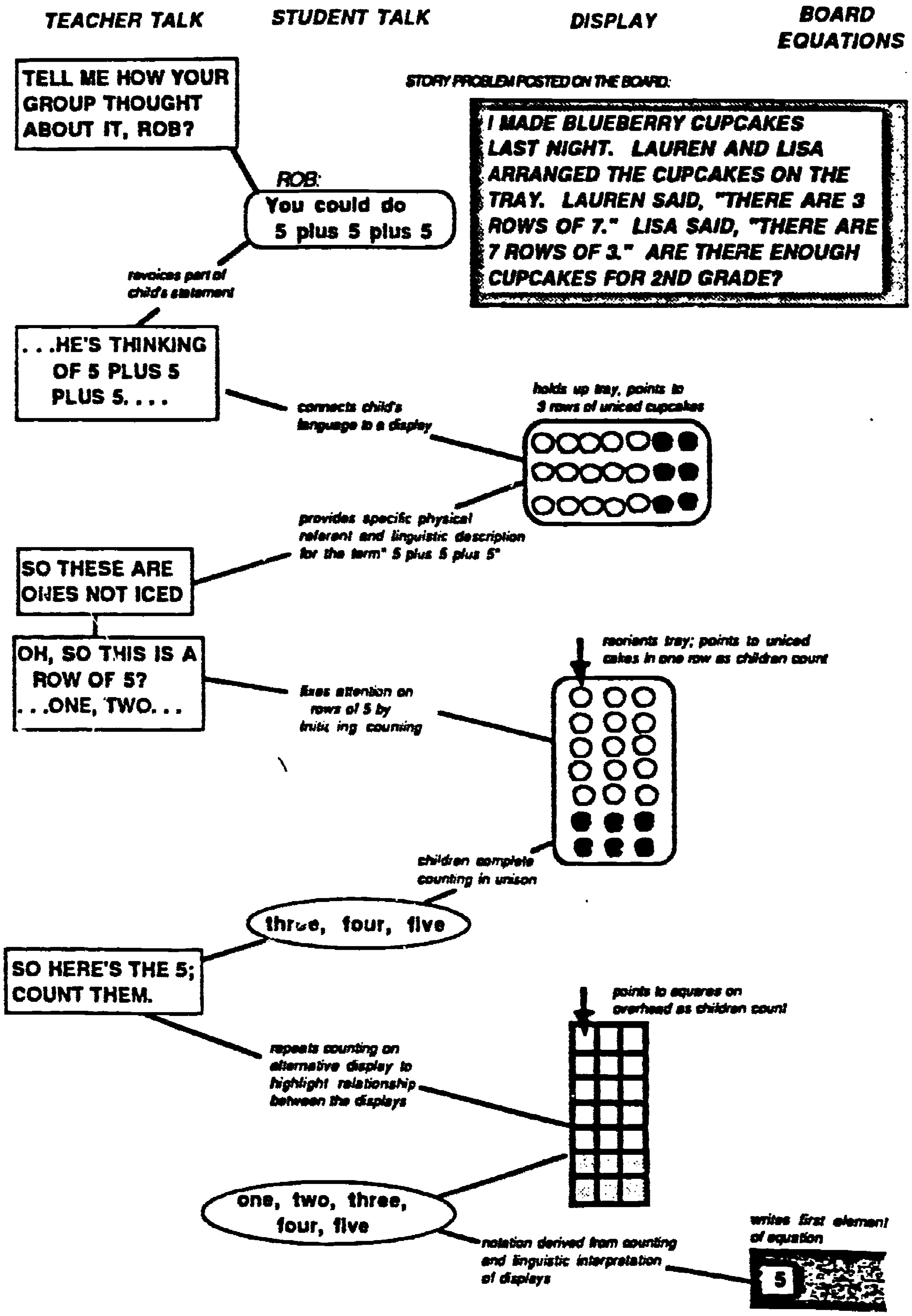


158 + 74 = 232
150 + 70 = 220
8 + 4 = 12
220 + 12 = 232

F



158 + 74 = 232
100 + 0 = 100
50 + 70 = 120
8 + 4 = 12
100 + 120 + 12 = 232
4


TEACHER TALK
STUDENT TALK
DISPLAY
BOARD EQUATIONS
TELL ME HOW YOUR GROUP THOUGHT ABOUT IT, ROB?
ROB:
You could do 5 plus 5 plus 5
revoices part of child's statement
. . .HE'S THINKING OF 5 PLUS 5 PLUS 5. . . .
connects child's language to a display
holds up tray, points to 3 rows of uniced cupcakes
provides specific physical referent and linguistic description for the term" 5 plus 5 plus 5"
SO THESE ARE ONES NOT ICED
OH, SO THIS IS A ROW OF 5? . . .ONE, TWO. . .
fixes attention on rows of 5 by initiating counting
reorients tray; points to uniced cakes in one row as children count
children complete counting in unison
three, four, five
SO HERE'S THE 5; COUNT THEM.
repeats counting on alternative display to highlight relationship between the displays
points to squares on overhead as children count
one, two, three, four, five
notation derived from counting and linguistic interpretation of displays
writes first element of equation
5

STORY PROBLEM POSTED ON THE BOARD:

I MADE BLUEBERRY CUPCAKES LAST NIGHT. LAUREN AND LISA ARRANGED THE CUPCAKES ON THE TRAY. LAUREN SAID, "THERE ARE 3 ROWS OF 7." LISA SAID, "THERE ARE 7 ROWS OF 3." ARE THERE ENOUGH CUPCAKES FOR 2ND GRADE?

Mary told her friend Tonya that she would give her 95 barrettes. Mary had 4 bags of barrettes and each bag had 9 barrettes. Does Mary have enough barrettes?

The class first developed an estimated answer. Then they were asked, "How many more does she need?" The solutions below were generated by different class groups.

Group 1 first solved for the number of barrettes by repeated addition. Then they decomposed 4 x 9 into 2 x 9 plus 2 x 9. Then they set up a missing addend problem, 36 + 59, which they solved by a combination of estimation and correction.

Group 2 set up a subtraction equation and then developed a solution that used a negative partial result.

Group 4 began with total number of barrettes needed and subtracted out the successive bags of 9.

Est. $4 \times 10 = \boxed{90}$ NO

1-24-90

#1 $9+9+9+9=36 \rightarrow 4 \times 9 = 36$

$2 \times 9 = 18$

$2 \times 9 = 18$

$18 + 18 = 36$

$36 + \overset{\boxed{60}}{\underline{59}} \overset{?}{=} 95$

$36 + \boxed{60} = 96$

$96 - 1 = 95$

$60 - 1 = 59$

#2 $95 - 36 = \underline{59}$

$90 - 30 = 60$

$5 - 6 = -1$

$60 - 1 = 59$

#4 $95 - 9 = 86$

$86 - 9 = 77$

$77 - 9 = 68$

$68 - 9 = \boxed{59}$

Domain	Specific Content
Reading/Writing Numerals	0-9,999
Set Counting	0-9,999
Addition	2- and 3-digit regrouping, Basic Facts 20
Subtraction	2-digit renaming, Basic Facts 20
Word Problems	Addition, Subtraction, Multiplication
Problem Solving	Work backward, Solve an easier problem, Patterns
Estimation	Quantities, Strategies, Length
Ratio/Proportion	Scaling up, Scaling down
Statistics/Probability	Scaling up, Scaling down, Spinner (1/4), Dice (1/16), 3 graphs
Multiplication	Array (2, 4 tables), Allocation, Equal groupings
Division	Oral problems involving sharing sets equally
Measurement	Arbitrary units
Decimals	Money
Fractions	Parts of whole, Parts of set, Equivalent pieces
Telling Time	To hour, To half hour
Geometry	Rectangle, square (properties)
Negative Integers	Ones, tens

Table 1

Constraint/Transformation Productions

	Both Parts Known (addition)	Whole and One Part Known (subtraction)
Change-Part-and-Part		
Increase in Part-X	CT1 IF PART-X --> PART-X + n PART-Y --> PART-Y - n THEN WHOLE --> WHOLE	CT2 IF PART-X --> PART-X + n WHOLE --> WHOLE THEN PART-Y --> PART-Y - n
Decrease in Part-X	CT3 IF PART-X --> PART-X - n PART-Y --> PART-Y + n THEN WHOLE --> WHOLE	CT4 IF PART-X --> PART-X - n WHOLE --> WHOLE THEN PART-Y --> PART-Y + n
Change-Part-and-Whole		
Increase in Part-X	CT5 IF PART-X --> PART-X + n PART-Y --> PART-Y THEN WHOLE --> WHOLE + n	CT6 IF PART-X --> PART-X + n WHOLE --> WHOLE + n THEN PART-Y --> PART-Y
Decrease in Part-X	CT7 IF PART-X --> PART-X - n PART-Y --> PART-Y THEN WHOLE --> WHOLE - n	CT8 IF PART-X --> PART-X - n WHOLE --> WHOLE - n THEN PART-Y --> PART-Y
No Change in Part-X	--	CT9 IF PART-X --> PART-X WHOLE --> WHOLE + n THEN PART-Y --> PART-Y + n
	--	CT10 IF PART-X --> PART-X WHOLE --> WHOLE - n THEN PART-Y --> PART-Y - n

Table 2. Layers of Mathematical Knowledge

Mathematics of:	Objects of reasoning	Linguistic terms	Operations
protoquantities	physical material	much, many, more, less, big, small, etc.	increase, decrease, combine, separate, compare, order.
quantities	measured physical material	n objects, n inches, n pounds, etc. Add, take away, divide	increase & decrease quantified sets by specific numbers of objects; increase & decrease measured amounts of material by measured amounts combine & partition quantified sets or measured amounts repeatedly add or remove a measured amount or set divide a set or measured amount into equal shares
numbers	specific numbers	n more than, n times, plus n, minus n, times n, n plus m, n divided by m	actions of adding, subtracting, multiplying, dividing applied to specific numbers
operators	numbers in general, operations, variables	addition, subtraction, multiplication, division, difference, equivalence, times greater than, times less than, 1/nth of	commute, associate, distribute, compose, decompose